# Bornes supérieures pour les valeurs propres des opérateurs naturels sur des variétés Riemanniennes compactes

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*À ma Mère, et à mon Père*

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# Résumé

Le but de cette thèse est de trouver des bornes supérieures pour les valeurs propres des opérateurs naturels agissant sur les fonctions d'une variété compacte $(M, g)$. Nous étudions l'opérateur de Laplace–Beltrami et des opérateurs du type laplacien. Dans le cas de l'opérateur de Laplace–Beltrami, deux aspects sont étudiés.
Le premier aspect est d'étudier les relations entre la géométrie intrinsèque et les valeurs propres du laplacien. Nous obtenons des bornes supérieures ne dépendant que de la dimension et d'un invariant conforme qui s'appelle le volume conforme minimal. Asymptotiquement, ces bornes sont en cohérence avec la loi de Weyl. Elles améliorent également les résultats de Korevaar et de Yang et Yau. La preuve repose sur la construction d'une famille convenable de domaines disjoints fournissant des supports pour une famille de fonctions tests. Cette méthode est puissante et intéressante en soi.
Le deuxième aspect est d'étudier la relation entre la géométrie extrinsèque et les valeurs propres du laplacien agissant sur des sous-variétés compactes de l'espace euclidien $\mathbb{R}^N$ ou de l'espace projectif complexe $\mathbb{C}P^N$. Nous étudions un invariant extrinsèque qui s'appelle l'indice d'intersection étudié par Colbois, Dryden et El Soufi. Pour des sous-variétés compactes de $\mathbb{R}^N$, nous généralisons leurs résultats et obtenons des bornes supérieures qui sont stables l'effet de petites perturbations. Pour des sous-variétés de $\mathbb{C}P^N$, nous obtenons une borne supérieure ne dépendant que du degré des sous-variétés et qui est optimale pour la première valeur propre non nulle.
Comme autre application de la méthode introduite, nous obtenons une borne supérieure pour des valeurs propres du problème de Steklov sur des sous-domaines à bord $C^1$ d'une variété riemannienne complete, en termes du rapport isopérimétrique du domaine, et du volume conforme minimal. Une modification de notre méthode donne des bornes supérieures pour les valeurs propres des opérateurs de Schrödinger en termes du volume conforme minimal et de l'intégrale du potentiel. Nous obtenons également les bornes supérieures pour les valeurs propres du laplacien de Bakry–Émery dépendant d'invariants conformes.

**Mots clés :** Opérateur de Laplace, opérateur de Schrödinger, opérateur de Laplace Barky-Émery, valeurs propres, borne suppérieure, volume confrome minimal, nombre d'intersection moyenne.

# Abstract

The purpose of this thesis is to find upper bounds for the eigenvalues of natural operators acting on functions on a compact Riemannian manifold $(M, g)$ such as the Laplace–Beltrami operator and Laplace-type operators. In the case of the Laplace–Beltrami operator, two aspects are investigated:
The first aspect is to study relationships between the intrinsic geometry and eigenvalues of the Laplace–Beltrami operator. In this regard, we obtain upper bounds depending only on the dimension and a conformal invariant called *min-conformal volume*. Asymptotically, these bounds are consistent with the Weyl law. They improve previous results by Korevaar and Yang and Yau. The proof relies on the construction of a suitable family of disjoint domains providing supports for a family of test functions. This method is powerful and interesting in itself.
The second aspect is to study the interplay of the extrinsic geometry and eigenvalues of the Laplace–Beltrami operator acting on compact submanifolds of $\mathbb{R}^N$ and of $\mathbb{C}P^N$. We investigate an extrinsic invariant called the *intersection index* studied by Colbois, Dryden and El Soufi. For compact submanifolds of $\mathbb{R}^N$, we extend their results and obtain upper bounds which are stable under small perturbation. For compact submanifolds of $\mathbb{C}P^N$, we obtain an upper bound depending only on the *degree* of submanifolds and which is sharp for the first eigenvalue.
As a further application of the introduced method, we obtain an upper bound for the eigenvalues of the Steklov problem in a domain with $C^1$ boundary in a complete Riemannian manifold in terms of the isoperimetric ratio of the domain and the min-conformal volume. A modification of our method also lead to have upper bounds for the eigenvalues of Schrödinger operators in terms of the min-conformal volume and integral quantity of the potential. As another application of our method, we obtain upper bounds for the eigenvalues of the Bakry–Émery Laplace operator depending on conformal invariants and properties of the weighted function.

**Keywords :** Laplace-Beltrami operator, Schrödinger operator, Bakry–Émery Laplace operator, eigenvalue, upper bound, min-conformal volume, mean intersection index.

# Contents

# Introduction

## Motivations et rappels historiques

La géométrie spectrale étudie les relations entre les valeurs propres d'opérateurs différentiels naturels sur des variétés, comme l'opérateur de Laplace-Beltrami, l'opérateur de Schrödinger, etc., et d'autres invariants géométriques de cette variété. L'objectif de cette thèse est d'étudier ces relations, en particulier de trouver des bornes supérieures pour les valeurs propres d'opérateurs naturels en fonction d'invariants géométriques.
Soit $(M, g)$ une variété riemannienne compacte et orientable de dimension $m$. L'opérateur de Laplace–Beltrami $\Delta_g$ est défini par $\Delta_g = -\mathrm{div}\nabla_g$. Il est bien connu que le spectre de l'opérateur de Laplace-Beltrami agissant sur les fonctions est discret et consiste en une suite non décroissante $\{\lambda_k(M, g)\}_{k=1}^{\infty}$ de valeurs propres de multiplicité finie.

La loi de Weyl donne le comportement asymptotique des valeurs propres de l'opérateur de Laplace–Beltrami :

$$\lambda_k(M, g) \left(\frac{\mu_g(M)}{k}\right)^{\frac{2}{m}} \sim \alpha_m, \quad k \to \infty$$

où $\mu_g$ désigne la mesure riemannienne associée à $g$, $\alpha_m = 4\pi^2\omega_m^{-\frac{2}{m}}$ et $\omega_m$ est le volume de la boule unité standard dans $\mathbb{R}^m$.
L'une des questions centrales que l'on étudie est de trouver des bornes supérieures pour le spectre de l'opérateur de Laplace–Beltrami qui soient consistantes avec la loi de Weyl dans le sens suivant :
Pour toute variété compacte $(M, g)$ de dimension $m$, on cherche une quantité $C$ telle que, pour tout $k \in \mathbb{N}^*$, on a

$$\lambda_k(M, g) \left(\frac{\mu_g(M)}{k}\right)^{\frac{2}{m}} \leq C. \tag{1}$$

D'abord on peut se demander si l'on peut trouver une telle constante ne dépendant que de la dimension. Il se trouve que la première valeur propre non nulle normalisée, $\lambda_2(M, g)\mu_g(M)^{\frac{2}{m}}$ ne peut pas être bornée supérieurement seulement en fonction de $m$ (voir par exemple [11], [13], [30], [39]). Ainsi, les bornes supérieures devront dépendre d'autres invariants géométriques.
Afin d'être asymptotiquement compatible avec la loi de Weyl, nous recherchons des bornes supérieures qui dépendent de la dimension et d'autres données géométriques telles qu'elles ne dépendent que de la dimension lorsque $k$ tend vers l'infini. Un exemple typique de tels

majorants est donné par l'inégalité de Buser [6] (voir aussi [16] et [28]) :
Il existe une constante $\alpha_m$, ne dépendant que de $m$, telle que pour toute variété compacte $(M,g)$ de dimension $m$ vérifiant $\mathrm{Ricci}_g(M) \geq -\kappa^2(m-1)$ pour $\kappa \in \mathbb{R}$ et pour tout $k \in \mathbb{N}^*$ :

$$\lambda_k(M,g)\left(\frac{\mu_g(M)}{k}\right)^{2/m} \leq \frac{(m-1)^2}{4}\kappa^2\left(\frac{\mu_g(M)}{k}\right)^{2/m} + \alpha_m. \tag{2}$$

On peut constater que ces bornes supérieures sont compatibles avec la loi de Weyl puisque le terme de droite de l'inégalité ne dépend asymptotiquement que de la dimension et pas d'autres invariants géométriques.
Il existe de nombreux résultats donnant des contrôles géométriques du spectre. Un autre exemple concerne la recherche de bornes supérieures dans une classe conforme. En ce qui concerne la première valeur propre non nulle $\lambda_2$, El Soufi et Ilias [18] (voir aussi [20]) ont démontré une inégalité similaire à l'inégalité (1) pour la première valeur propre $\lambda_2$ du laplacien avec une constante $C_m([g])$ dépendant de la classe conforme $[g]$ de la métrique $g$ (en fait, le volume conforme introduit par Li et Yau [29] qui prouve le même résultat en dimension 2). Yang et Yau [43] (voir aussi [29]) ont démontré l'inégalité (1) pour $\lambda_2$ avec une constante ne dépendant que du genre de la surface. En 1993, Korevaar [27] a généralisé ces résultats aux valeurs propres d'ordres supérieurs. En particulier, pour les surfaces, il donne une réponse affirmative à une conjecture de Yau [44]. Plus précisément, Korevaar obtient la borne supérieure conforme suivante :
(i) Si $(M^m,g)$ est une variété riemannienne compacte de dimension $m$, alors pour tout $k \in \mathbb{N}^*$,

$$\lambda_k(M,g)\left(\frac{\mu_g(M)}{k}\right)^{\frac{2}{m}} \leq c_m([g]), \tag{3}$$

où $c_m([g])$ est une constante ne dépendant que de la classe conforme $[g]$ de la métrique $g$.
(ii) Si $(\Sigma_\gamma, g)$ est une surface compacte orientable de genre $\gamma$, alors pour tout $k \in \mathbb{N}^*$,

$$\lambda_k(\Sigma_\gamma,g)\frac{\mu_g(\Sigma_\gamma)}{k} \leq C(\gamma+1), \tag{4}$$

où $C$ est une constante universelle. Rappelons qu'il est impossible d'avoir une borne supérieure ne dépendant que de la topologie en dimension supérieure à cause d'un résultat de Colbois et Dodziuk [11].
Cependant, les inégalités (3) et (4) ne sont pas asymptotiquement consistantes avec la loi de Weyl dans le sens que nous avons présenté au début. En effet, la constante du côté droit de cette inégalité dépend toujours soit de la classe conforme de la métrique, soit du genre, lorsque $k$ tend vers l'infini. Maintenant, on se pose deux questions naturelles :

**Question 1.** *Est-ce que nous pouvons donner une description explicite de la constante conforme de l'inégalité (3) ?*

**Question 2.** *Est-ce que nous pouvons obtenir des bornes supérieures ne dépendant que de la dimension et pas de la géometrie ?*

Répondre à ces deux questions est le but principal du premier chapitre.

Un autre aspect de la géométrie spectrale est d'étudier la relation entre la géométrie extrinsèque de sous-variétés et le spectre du laplacien. L'un des invariants extrinsèques bien connu est le champ vectoriel de la courbure moyenne d'une sous-variété. À ce propos, on peut mentioner l'inégalité de Reilly [34] pour des sous–variétés immergées de dimension $m$ de $\mathbb{R}^N$

$$\lambda_2(M) \leq \frac{m}{\mathrm{Vol}(M)} \|H(M)\|_2^2,$$

où $\|H(M)\|_2$ est la norme $L^2$ du champ vectoriel de la courbure moyenne de $M$. Pour les valeurs propres d'ordre supérieur, il découle des résultats de El Soufi, Harrell et Ilias [17] et de la formule de récurrence de Cheng et Yang [8, Corollary 2.1] que pour tout $k \in \mathbb{N}^*$,

$$\lambda_k(M) \leq R(m)\|H(M)\|_\infty^2 \, k^{2/m},$$

où $\|H(M)\|_\infty$ est la norme $L^\infty$ de $H(M)$ et $R(m)$ est une constante ne dépendant que de $m$. Nous sommes intéressés à des invariants extrinsèques qui ne dépendent pas des dérivées de la métrique, excluant par exemple la courbure.
Dans [12], Colbois, Dryden et El Soufi ont étudié la relation entre les valeurs propres de l'opérateur de Laplace–Beltrami et un invariant extrinsèque des sous-variétés de $\mathbb{R}^N$, appellé *l'indice d'intersection* $i(M)$, défini comme suit : Pour une sous–variété immergée compacte $M$ de dimension $m$ de $\mathbb{R}^{m+p}$, $p > 0$, *l'indice d'intersection* est donné par

$$i(M) = \sup_{\Pi} \sharp(M \cap \Pi),$$

où $\Pi$ varie dans l'ensemble des $p$-plans qui sont transverses à $M$ ; si $M$ n'est pas plongée, on compte les points de $M$ plusieurs fois en fonction de leurs multiplicités. Il s'avère que l'inégalité (1) est vraie pour les valeurs propres du laplacian sur une sous-variété de $\mathbb{R}^N$ avec une constante $C$ ne dépendant que de l'indice d'intersection $i(M)$ et de la dimension de la sous-variété.

$$\lambda_k(M) \left(\frac{\mathrm{Vol}(M)}{k}\right)^{2/m} \leq c(m) i(M)^{2/m}.$$

Une conséquence remarquable de ce résultat concerne les sous-variétés algébriques. Il donne une borne supérieure ne dépendant que du degré de la variété pour les valeurs propres du laplacien sur les variétés algébriques réelles et compactes. Notons que ces résultats ne sont pas stables par des *petites perturbations*. On peut maintenant poser les questions suivantes :

**Question 3.** *Est-ce que nous pouvons remplacer l'indice d'intersection par une version modifiée qui soit stable par des "petites perturbations" ?*

**Question 4.** *Est-ce que nous pouvons obtenir un invariant algébrique, comme le degré, pour une borne supérieure des valeurs propres du laplacien agissant sur des sous-variétés complexes de l'espace projective complexe ?*

Notre objectif dans le deuxième chapitre est de répondre à ces questions. Nous généralisons le travail de Colbois, Dryden et El Soufi dans deux directions. La première consiste à remplacer l'indice d'intersection $i(M)$ par des invariants du même type qui sont stables par "petites perturbations". La deuxième direction concerne des sous-variétés complexes de l'espace projectif complexe $\mathbb{C}P^N$.

Finalement, on considère l'opérateur de Schrödinger agissant sur une variété riemannienne compacte $(M, g)$. Les valeurs propres de l'opérateur de Schrödinger $L = \Delta_g + q$, où

$q$ est une fonction continue sur $M$, constituent une suite non décroissante non-bornée de nombres réels. Par la caractérisation variationnelle, nous constatons que la première valeur propre est contrôlée par la moyenne du potentiel $q$. Pour la deuxième valeur propre $\lambda_2(L)$, une borne supérieure en termes de la moyenne du potentiel $q$ et d'un invariant conforme a été obtenue par El Soufi et Ilias [18, théorème 2.2] :

$$\lambda_2(\Delta_g + q) \leq m \left( \frac{V_c([g])}{\mu_g(M)} \right)^{\frac{2}{m}} + \frac{\int_M q d\mu_g}{\mu_g(M)},$$

où $V_c([g])$ est le volume conforme défini par Li et Yau [29] qui ne dépend que de la classe conforme $[g]$ de la métrique $g$.
Pour une surface riemannienne $(\Sigma_\gamma, g)$ de genre $\gamma$, le volume conforme peut se remplacer par le genre $\gamma$ de la surface. Maintenant, on se pose la question suivante.

**Question 5.** *Est-ce que nous pouvons contrôler les valeurs propres de l'opérateur Schrödinger $L$ en terms de la moyenne du potentiel $q$ et des invariants géométriques de $M$ ?*

Cette question a été étudiée par Grigor'yan, Netrusov et Yau [24]. Ils ont donné une réponse affirmative à la question lorsque l'opérateur de Schrödinger $L$ est positif (voir (3.5)). En général, ils ont obtenu des bornes supérieures en termes de quantités intégrales en fonction du potentiel $q$ (voir les inégalités (3.4), (3.5)) muni de certaines conditions sur la métrique ; cependant, leurs résultats ne sont pas consistants avec la loi de Weyl concernant la puissance de $k$, sauf en dimension 2. Dans ce cas, la borne supérieure qu'ils ont obtenue [24, théorème 5.4], dépend du genre et de quantités intégrales fonction du potentiel $q$.

Dans le troisième chapitre, nous obtenons des bornes supérieures qui généralisent les résultats de [24] sans imposer des contraintes géométriques. Ces bornes sont asymptotiquement compatibles avec la loi de Weyl.

Pour aller plus loin, nous étudions le laplacien à poids ou laplacien de Bakry–Émery $\Delta_\phi$ et nous obtenons des bornes supérieures pour ces valeurs propres. On peut poser la question suivante

**Question 6.** *Quelle est la relation entre les valeurs propres du laplacien de Bakry–Émery et des invariantes géométriques associés à la mesure pondérée ?*

Le dernier paragraphe du troisième chapitre est consacré à l'étude des valeurs propres du laplacien de Bakry–Émery et traite la question 6.

## Présentation des résultats

Dans la suite, nous supposerons toujours que nos variétés sont orientables. Dans ce qui suit, nous présentons les théorèmes principaux de cette thèse qui répondent également aux questions posées dans le paragraphe précédent.

**Chapitre 1.** Dans le même esprit que les résultats de Korevaar, notre but est d'obtenir des bornes supérieures conformes et qui sont également consistantes avec la loi de Weyl. Notre approche consiste à prendre une constante $C$ de l'inégalité (1) comme somme de

deux quantités : l'une dépend de la classe conforme $[g]$ ou du genre $\gamma$ pour les surfaces et tend vers zéro lorsque $k$ tend vers infini, et l'autre ne dépend que de la dimension. Afin d'énoncer notre résultat, nous avons besoin d'introduire l'invariant conforme suivant.
Soit $(M,g)$ une variété riemannienne compacte de dimension $m$. Nous définissons son *volume conforme minimal* comme suit :

$$V([g]) = \inf\{\mu_{g_0}(M) \ : \ g_0 \in [g], \ \mathrm{Ricci}_{g_0} \geq -(m-1)\}.$$

Le théorème suivant donne une réponse affirmative aux questions 1 et 2.

**Théorème 1.** *Pour tout entier $m \geq 2$, il existe deux constantes $A_m$ et $B_m$ telles que, pour toute variété riemannienne compacte $(M,g)$ de dimension $m$ et tout $k \in \mathbb{N}^*$, nous avons*

$$\lambda_k(M,g)\left(\frac{\mu_g(M)}{k}\right)^{\frac{2}{m}} \leq A_m\left(\frac{V([g])}{k}\right)^{\frac{2}{m}} + B_m.$$

*En particulier, en dimension 2, il existe des constantes universelles $A$ et $B$ telles que pour toute surface riemannienne compacte $\Sigma_\gamma$ de genre $\gamma$ et tout $k \in \mathbb{N}^*$, nous avons*

$$\lambda_k(\Sigma_\gamma, g)\frac{\mu_g(\Sigma_\gamma)}{k} \leq A\frac{\gamma}{k} + B. \tag{5}$$

L'inégalité (5) donne une borne supérieure pour le spectre topologique introduit par Colbois et El Soufi [13], et qui peut être comparée avec la borne inférieure qu'ils ont obtenue [13, page 341].

L'avantage principal de notre approche est de nous permettre de retrouver l'inégalité (1) avec une constante ne dépendant que de la dimension, pour tout entier $k$ supérieur à une constante ne dépendant que de $[g]$ ou de $\gamma$. Les inégalités suivantes sont les conséquences directes du théorème 1 :
Il existe une constante $B' > 0$ et, pour chaque $m \geq 2$, une constante $B'_m > 0$, telles que les propriétes suivantes sont satisfaites :

(1) pour toute variété riemannienne compacte $(M,g)$ de dimension $m \geq 2$, il existe un entier $k_0([g])$ ne dépendant que de la classe conforme de $g$, tel que pour tout $k \geq k_0([g])$,

$$\lambda_k(M,g)\left(\frac{\mu_g(M)}{k}\right)^{\frac{2}{m}} \leq B'_m;$$

(2) pour toute surface riemannienne compacte $(\Sigma_\gamma, g)$ de genre $\gamma$, il existe un entier $k_0(\gamma)$ ne dépendant que de $\gamma$, tel que pour tout $k \geq k_0(\gamma)$,

$$\lambda_k(\Sigma_\gamma, g)\frac{\mu_g(\Sigma_\gamma)}{k} \leq B'.$$

Dans le dernier paragraphe du chapitre 1, nous étudions le problème des valeurs propres de Steklov comme une autre illustration de notre méthode. Dans un article récent, Girouard et Polterovich [22, théorème 1.2] (aussi voir [14], [19] et [21]) montrent l'inégalité suivante pour les valeurs propre de Steklov d'une surface riemannienne compacte $(\Sigma_\gamma, g)$ de genre $\gamma$ et de $\kappa$ composantes de bord :

$$\sigma_k(\Sigma_\gamma)\ell_g(\partial\Sigma_\gamma) \leq 2\pi(\gamma+\kappa)k,$$

où $\ell_g(\partial\Sigma)$ est la longueur du bord. Pour une surface riemannienne compacte, nous obtenons

**Théorème 2.** *Soient $(\Sigma_\gamma, g)$ une surface riemannienne compacte de genre $\gamma$, et $\Omega$ un domaine de $\Sigma_\gamma$. Alors*

$$\sigma_k(\Omega)\ell_g(\partial\Omega) \leq A\gamma + Bk, \tag{6}$$

*où $A$ et $B$ sont des constantes universelles.*

Il faut remarquer que nos constantes sont loin d'être optimales. On ne peut pas espérer obtenir des constantes optimales avec notre méthode. En dimension supérieure, nous obtenons une borne supérieure ne dépendant que de la classe conforme de la métrique et du rapport isopérimétrique (voir théorème 1.4.1) que l'on peut comparer avec les résultats de Colbois, El Soufi et Girouard [14].

**Chapitre 2.** Dans ce chapitre, nous généralisons les résultats de [12] et donnons également des réponses aux questions 3 et 4. Nous définissons des versions modifiées de l'indice d'intersection $i(M)$ étudié dans [12] comme suit : Soit $G$ le grassmannienne des sous-espaces vectoriels de dimension $k$ dans $\mathbb{R}^{m+p}$ avec une mesure de Haar de mesure totale 1. Soient $0 < \varepsilon < 1$ et $D$ un domaine ouvert quelconque de $M$ tels que $M \setminus D$ soit une variété lisse avec bord lisse et $\mathrm{Vol}(D) \leq \varepsilon \mathrm{Vol}(M)$. Nous notons $M \setminus D$ par $M_\varepsilon^D$ et $\sup_{P\perp H} \sharp(M_\varepsilon^D \cap P)$ par $i_H(M_\varepsilon^D)$, où $P$ est un $p$-plan affine orthogonal à $H$. La *$\varepsilon$-moyenne de l'indice d'intersection* est définie par :

$$\bar{\imath}^\varepsilon(M) := \inf_D \int_G i_H(M_\varepsilon^D)dH,$$

où l'infimum est pris sur tous les domaines $D$ tels que la mesure de $D$ est plus petite que $\varepsilon\mathrm{Vol}(M)$ et $M \setminus D$ est un variété lisse avec bord lisse.
Pareillement, pour $r > 0$, nous définissons *l'indice d'intersection $(r, \varepsilon)$-local* par :

$$\bar{\imath}_r^\varepsilon(M) := \inf_D \sup_{x \in M_\varepsilon^D} \int_G i_H(M_\varepsilon^D \cap B(x,r))dH,$$

où $B(x,r) \subset \mathbb{R}^{m+p}$ est une boule euclidienne centrée en $x$ et de rayon $r$, et $D$ varie sur les domaines tels que la mesure de $D$ est plus petite que $\varepsilon\mathrm{Vol}(M)$ et $M \setminus D$ est une variété lisse avec bord lisse.
Il est intéressant de remarquer que ces nouvelles notions sont stables par des *perturbations $\varepsilon$-petite*[1]. Nous obtenons le théorème suivant

**Théorème 3.** *Il existe des constantes positives $c_m, \alpha_m$ et $\beta_m$ ne dépendant que de $m \geq 2$ telles que pour toute sous–variété immergée $M$ de dimension $m$ de $\mathbb{R}^{m+p}$ et tout $k \in \mathbb{N}^*$ et $\varepsilon > 0$, nous avons*

$$\lambda_k(M)\, Vol(M)^{2/m} \leq c_m \frac{\bar{\imath}^\varepsilon(M)^{2/m}}{(1-\varepsilon)^{1+2/m}} k^{2/m},$$

[1]Soit $\varepsilon$ un nombre positif, $\varepsilon < 1$. La perturbation $\varepsilon$-petite est n'importe quelle perturbation dans un domaine $D \subset M$ avec mesure plus petite ou égale à $\varepsilon$Vol(M).

$$\lambda_k(M) \leq \alpha_m \frac{1}{(1-\varepsilon)r^2} + \beta_m \frac{\overline{i}_r^{\varepsilon}(M)^{2/m}}{(1-\varepsilon)^{1+2/m}} \left(\frac{k}{Vol(M)}\right)^{2/m}.$$

Le théorème suivant répond à la question 4

**Théorème 4.** *Soit $M^m$ une variété complexe de dimension $m$ admettant une immersion holomorphe $\phi : M \to \mathbb{C}P^N$. Alors pour tout $k \in \mathbb{N}^*$ nous avons*

$$\lambda_{k+1}(M, \phi^* g_{FS}) \leq 2(m+1)(m+2)k^{\frac{1}{m}} - 2m(m+1),$$

*où $g_{FS}$ est la métrique de Fubini–Study sur $\mathbb{C}P^N$.*

En particulier, on a l'inégalité ci-dessus pour toute sous-variété complexe de $\mathbb{C}P^N$. Pour une sous-variété complexe $M$ de $\mathbb{C}P^{m+p}$ de dimension complexe $m$, on a

$$\mathrm{Vol(M)} = \deg(M)\mathrm{Vol}(\mathbb{C}P^m),$$

où $\deg(M)$ est le nombre de l'intersection de $M$ avec un plan projectif de dimension $p$ en position générique. De plus, on peut considérer $M$ comme l'ensemble des racines communes d'un nombre fini de polynômes homogènes et irréductibles. Alors, $\deg(M)$ est le produit des degrés des polynômes qui définissent $M$ (regarder par exemple [23, pages 171-172]). Par conséquent, on peut écrire l'inégalité (2.5) comme suit :

$$\lambda_{k+1}(M, \phi^* g_{FS})\mathrm{Vol}(M)^{\frac{1}{m}} \leq C(m) \deg(M)^{\frac{1}{m}} k^{\frac{1}{m}}.$$

On voit que la puissance de $k$ est consistante avec la loi de Weyl.

**Remarque 1.** *Pour $k = 1$, on a*

$$\lambda_2(M, \phi^* g_{FS}) \leq 4(m+1),$$

*qui est une inégalité optimale car l'égalité a lieu si $M = \mathbb{C}P^m$. Cette borne supérieure optimale a été obtenue par Bourguignon, Li et Yau [4, page 200] (regarder aussi [1]) avec l'hypothèse que l'immersion holomorphe $\phi$ devait être une immersion pleine. Le théorème 2.1.2 nous donne une preuve de cette inégalité optimale sans cette hypothèse.*

**Chapitre 3.** Le premier paragraphe du troisième chapitre concerne la question 5. Nous obtenons l'extension suivante des résultats de [24] pour l'opérateur de Schrödinger $L = \Delta_g + q$, $q \in C^0(M)$.

**Théorème 5.** *Il existe des constantes $\alpha_m \in (0,1), B_m$ et $C_m$ ne dépendant que de $m \geq 2$ telles que pour toute variété riemannienne compacte $(M, g)$ de dimension $m$, tout potentiel $q \in C^0(M)$ et tout $k \in \mathbb{N}^*$, nous avons*

$$\lambda_k(\Delta_g + q) \leq \frac{\alpha_m^{-1} \int_M q^+ d\mu_g - \alpha_m \int_M q^- d\mu_g}{\mu_g(M)} + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{2/m} + C_m \left(\frac{k}{\mu_g(M)}\right)^{2/m},$$

*où $V([g])$ est le volume conforme minimal et $q^{\pm} = \max\{|\pm q|, 0\}$.*
*En particulier, quand le potentiel $q$ est non-négatif, on a*

$$\lambda_k(\Delta_g + q) \leq A_m \frac{1}{\mu_g(M)} \int_M q d\mu_g + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}},$$

*où $A_m = \alpha_m^{-1}$.*

De plus, quand l'opérateur de Schrödinger $L$ est positif nous obtenons

**Théorème 6.** *Il existe des constantes $A_m > 1$, $B_m$ et $C_m$ ne dépendant que de $m \geq 2$ telles que si $L = \Delta_g + q$, $q \in C^0(M)$ est un opérateur positif, alors pour toute variété riemannienne compacte $(M, g)$ de dimension $m$ et tout $k \in \mathbb{N}^*$ nous obtenons*

$$\lambda_k(\Delta_g + q) \leq A_m \frac{\int_M q d\mu_g}{\mu_g(M)} + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}.$$

Cela donne le corollaire suivant pour l'opérateur de Schrödinger $L = \Delta_g + q$, $q \in C^0(M)$.

**Corollaire 1.** *Il existe des constantes $A_m > 1$, $B_m$ et $C_m$ ne dépendant que de $m \geq 2$ telles que pour toute variété riemannienne compacte $(M, g)$ de dimension $m$, tout potentiel $q \in C^0(M)$ et tout $k \in \mathbb{N}^*$, nous obtenons*

$$\lambda_k(\Delta_g + q) \leq A_m \frac{1}{\mu_g(M)} \int_M q d\mu_g + (1 - A_m)\lambda_1(\Delta_g + q) + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}.$$

Pour des surfaces riemanniennes, on peut remplacer $V([g])$ par un invariant topologique ne dépendant que du genre. Tous les bornes supérieures ci-dessus sont consistantes avec la loi de Weyl. Comme application des résultats ci-dessus, nous obtenons la borne supérieure suivante pour les valeurs propres du laplacien à poids ou laplacien de Bakry–Émery $\Delta_\phi$. Prenons une variété riemannienne $(M, g)$ et une fonction $\phi \in C^2(M)$, le laplacien à poids associé $\Delta_\phi$ est défini comme suit :

$$\Delta_\phi = \Delta_g + \nabla_g \phi \cdot \nabla_g.$$

**Théorème 7.** *Il existe des constantes $A_m > 1$, $B_m$ et $C_m$ ne dépendant que de $m \geq 2$, telles que pour toute variété riemannienne compacte $(M, g)$ de dimension $m$, tout $\phi \in C^2(M)$ et tout $k \in \mathbb{N}^*$, nous obtenons*

$$\lambda_k(\Delta_\phi) \leq A_m \frac{1}{\mu_g(M)} \|\nabla_g \phi\|^2_{L^2(M)} + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}.$$

Rappelons qu'il y existe des résultats dans le cas des variétés complets (voir [33], [37], [41] et [42])) donnat des bornes supérieures pour la première valeur propre du laplacien de Bakry–Émery dépendant de la norme $L^\infty$ de $\nabla_g \phi$ et la borne inférieure de la courbure de Bakry–Émery–Ricci notée $\mathrm{Ricci}_\phi$. À ce propos, le théorème ci-dessus montre que dans le cas compact, la borne supérieur ne dépend que de la norme $L^2$ de $\nabla_g \phi$ et de la classe conforme de la métrique pour la première valeur propre non nulle. Il nous donne également une borne supérieure pour les valeurs propres d'ordre supérieur.
Les liens entre la géometrie de variété et la fonction pondérée $\phi$ sont présentés par une version modifiée de la courbure s'appellé la courbure de Bakry–Émery–Ricci. Ici, employons la méthode, utilisée avec succès dans le cas du laplacien, nous montrons le résultat suivant.

**Théorème 8.** *Soit $(M, g, \phi)$ une variété de Bakry-Émery avec $\partial_r \phi \geq -\sigma$ pour $\sigma \geq 0$. Alors, il existe des constantes $A(m)$ et $B(m)$ telles que pour tout $k \in \mathbb{N}^*$,*

$$\lambda_k(\Delta_\phi) \leq A(m) \max\{\sigma^2, 1\} \left(\frac{V_\phi([g])}{\mu_\phi(M, g)}\right)^{2/m} + B(m) \left(\frac{k}{\mu_\phi(M)}\right)^{2/m},$$

*où l'inégalité $\partial_r \phi > -\sigma$ est vérifiée, c'est-à-dire que pour tout $x \in M$ le gradient radial associé à $\phi$ est au moins égale à $-\sigma$.*

Il faut noter qu'en général, il n'est pas possible d'obtenir des bornes supérieures qui ne dépendent pas de $\phi$ (voir par exemple [37, section 2]). Cependant, pour des variétés compactes avec courbure de Bakry–Émery–Ricci non-negative, nous pouvons trouver des bornes supérieures qui ne dépendent pas de $\phi$ :

**Corollaire 2.** *Soit $(M, g, \phi)$ une variété de Bakry–Émery telle qu'il existe $g_0 \in [g]$ avec* $\mathrm{Ricci}_\phi(M, g) \geq 0$. *Alors, il existe une constante positive $A(m)$ ne dépendant que de la dimension telle que pour tout $k \in \mathbb{N}^*$*

$$\lambda_k(\Delta_\phi) \leq A(m) \left( \frac{k}{\mu_\phi(M)} \right)^{2/m}.$$

## Méthodes employées

Ici, nous donnons une idée des méthodes employées pour démontrer nos résultats. Nous commençons par la méthode essentielle de cette thèse. Dans le premier chapitre, nous donnons une nouvelle méthode pour trouver des bornes supérieures pour les valeurs propres du laplacien. Cette méthode est basée sur deux constructions données par Grigor'yan, Netrusov et Yau [24], et Colbois et Maerten [16]. Nous appelons ces deux constructions la construction GNY et la construction CM, respectivement.
Nous rappelons brièvement le point de départ pour trouver des bornes supérieures pour les valeurs propres du laplacien. Le théorème classique de min-max pour ces valeurs propres implique que si l'on a une famille $\{f_i\}_{i=1}^k$ des $k$ fonctions test de supports disjoints alors

$$\lambda_k(M, g) \leq \max_i \left\{ \frac{\int_M |\nabla_g f_i|^2 d\mu_g}{\int_M f_i^2 d\mu_g} \right\}.$$

Puis, le problème de l'estimation des valeurs propres se réduit premièrement, à trouver des condensateurs disjoints [2] qui seront les supports de nos fonctions test, et deuxièmement, à estimer $\frac{\int_M |\nabla_g f_i|^2 d\mu_g}{\int_M f_i^2 d\mu_g}$, $1 \leq i \leq k$, qui s'appelle le quotient de Rayleigh.
L'idée classique fournit une famille des boules comme condensateurs afin de construire une famille des fonctions plateau de support disjoints comme fonctions test. Cette construction, qui a été étudiée dans [6], [10] and [28], mène à des résultats typiques comme l'inégalité (2). Par la suite, Colbois et Maerten [16] ont introduit une construction plus élaborée qui permet par exemple d'avoir l'inégalité (2) sur des domaines bornés des variétés complètes. Nous remarquons que pour appliquer la méthode de Colbois et Maerten, il faut éviter d'avoir une concentration de la mesure i.e. les petites boules doivent avoir des petits volumes. Il suit que l'on ne peut pas avoir des bornes dans une classe conforme de métriques en utilisant cette construction. L'autre construction, donnée par Grigor'yan, Netrusov et Yau [24] (en reprenant la méthode introduite par Korevaar [27]), permet d'avoir des bornes supérieures via des capacitiés, moyennant certaines conditions sur les variété considérées. Elle consiste

[2] Chaque couple $(F, G)$ de l'ensemble ouvert dans $M$ tel que $F \subset G$ s'appelle un *condensateur*.

à trouver des bornes supérieures via capacités sans imposer ces conditions. Notre méthode consiste à étendre et à combiner la construction GNY et la construction CM.
Pour chaque condensateur, on définit une capacité par

$$\mathrm{cap}_g(F,G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^2 d\mu_g,$$

où $\mathcal{T} = \mathcal{T}(F,G)$ est l'ensemble de toutes les fonctions Lipschitz de supports compacts sur $M$ tel que supp $\varphi \subset \mathrm{G}^\circ = \mathrm{G} \setminus \partial\mathrm{G}$ et $\varphi \equiv 1$ dans un voisinage de $F$. Si $\mathcal{T}(F,G)$ est vide, alors $\mathrm{cap}_g(F,G) = +\infty$.
Soit $\{(F_i, G_i)\}_{i=1}^k$ une famille des condensateurs disjoints avec les propriétés suivantes

- $\mu_g(F_i) \geq \alpha$ ;
- $\mathrm{cap}(F_i, G_i) \leq \beta$.

Pour tout $\epsilon > 0$, on construit des fonctions test $f_i \in \mathcal{T}(F_i, G_i)$, $1 \leq i \leq k$, tels que

$$\int_M |\nabla_g f|^2 d\mu_g \leq \mathrm{cap}(F_i, G_i) + \epsilon.$$

Pour tout $\epsilon > 0$ et tout $1 \leq i \leq k$,

$$\frac{\int_M |\nabla_g f_i|^2 d\mu_g}{\int_M f_i^2 d\mu_g} \leq \frac{\beta + \epsilon}{\alpha},$$

et,

$$\lambda_k \leq \frac{\beta}{\alpha}.$$

Supposons que l'on a $F_1, \ldots, F_k$ tels que $\mu_g(F_i) \geq \frac{\mu_g(M)}{Ck}$ pour certaines constantes $C > 1$. Donc, $\lambda_k \leq C\beta \frac{k}{\mu_g(M)}$. Dans le cas où on cherche des bornes supérieures conformes, nous estimons la capacité en termes d'invariants conformes. En dimension 2, la capacité est un invariant conforme. En dimension supérieure $m > 2$, on peut estimer la $m$-capacité[3] qui est un invariant conforme. Par l'inégalité de Hölder nous obtenons

$$\mathrm{cap}(F_i, G_i) \leq \mathrm{cap}_{[g]}^{(m)}(F_i, G_i)^{2/m} \left( \frac{\mu_g(M)}{k} \right)^{1-2/m}.$$

Donc,

$$\lambda_k \left( \frac{\mu_g(M)}{k} \right)^{2/m} \leq C \mathrm{cap}_{[g]}^{(m)}(F_i, G_i)^{2/m}.$$

La construction GNY donne une condition suffisante pour contrôler la $m$-capacité par une constante ne dépendant que de $m$. Cette condition peut être reformulée comme suit. Si pour

[3]Pareillement, on peut définir la *$m$-capacité* comme

$$\mathrm{cap}_{[g]}^{(m)}(F,G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^m d\mu_g.$$

une famille $\mathcal{M}$ de variété compactes de dimension $m$, il existe une constante ne dépendant que de $m$ telle que pour toute $(M, g) \in \mathcal{M}$, certains $g_0 \in [g]$ et tout $x \in M$ et tout $r > 0$ on a

$$\mu_{g_0}(B(x,r)) \leq B(m)r^m,$$

alors on peut obtenir une borne supérieure ne dépendant que de $m$ pour les $m$-capacités construits par la construction GNY. Par conséquent, nous avons l'inégalité (1) avec une constante $C$ ne dépendant que de la dimension pour cette famille de variétés. Cette condition n'est pas satisfaite pour la famille de toutes variétés compactes. Par exemple, la famille de variété riemanniennes à courbure de Ricci non-négatives vérifient cette condition mais les variétés riemanniennes avec courbure de Ricci négative ne la satisfont pas. Cependant, on a toujours, localement, cette propriété. L'idée de notre construction est la suivante. Nous commençons avec la construction GNY qui procède par induction. Si chacun des condensateurs que nous construisons reste dans une boule de rayon $r_0$, où $r_0$ est donné, nous continuons le processus. Si le nombre de condensateurs construit avec cette méthode est égal à $k$, alors nous obtenons une borne supérieure pour la $m$-capacité qui ne dépend que de $m$. Sinon, nous pouvons appliquer la construction CM sur la complémentaire des condensateurs construits. Elle mène à une borne supérieure conforme pour la $m$-capacité. Nous remarquons que la construction CM toute seule ne permet pas d'avoir des bornes supérieures conformes ; néanmoins, avec cette approche de combiner convenablement ces deux constructions, nous avons une estimation pour la $m$-capacité comme suit :

$$\mathrm{cap}^{(m)}_{[g]}(F_i, G_i) \leq \beta_1 + \beta_2,$$

où $\beta_1$ est une constante ne dépendant que de $m$ et $\beta_2$ est un invariant conforme qui tend vers zéro lorsque $k$ tend vers l'infini. Nous résumons maintenant notre construction dans le théorème suivant.

**Théorème 9.** *Soit $(M^m, g, \mu)$ une variété riemannienne compacte avec une mesure non-atomique borélienne[4] $\mu$. Alors, il existe des constantes positives $c(m) \in (0,1)$ et $\alpha(m)$ ne dépendant que de la dimension telles que pour tout $k \in \mathbb{N}^*$, il existe une famille $\{(F_i, G_i)\}_{i=1}^k$ de condensateurs avec les propriétées suivants :*

*(I)* $\mu(F_i) > c(m)\frac{\mu(M)}{k}$,

*(II)* $\mathrm{cap}_g(F_i, G_i) \leq \frac{\mu_g(M)}{k}\left[\frac{1}{r_0^2}\left(\frac{V([g])}{\mu_g(M)}\right)^{2/m} + \alpha(m)\left(\frac{k}{\mu_g(M)}\right)^{2/m}\right]$.

*où* $r_0 = \frac{1}{1600}$.

**Remarque 2.** *Nous pouvons énoncer en partie le théorème ci-dessus dans le cadre général des espaces métriques mesurés (voir le théorème 1.2.1).*

Dans le troisième chapitre, nous utilisons une version modifiée de la méthode ci-dessus pour estimer les valeurs propres de l'opérateur de Schrödinger et du laplacien de Bakry–Émery. Dans le cas de l'opérateur de Schrödinger, en conséquence de la caractérisation

[4]La mesure $\mu$ sur l'espace métrique $(X, d)$ appelée non-atomique si et seulement si $\mu(\{x\}) = 0$ pour tout $x \in X$.

variationelle, on a

$$\lambda_k(\Delta_g + q) \leq \max_{i\in\{1,\ldots,k\}} \frac{\int_M |\nabla_g f_i|^2 d\mu_g + \int_M f_i^2 q d\mu_g}{\int_M f_i^2 d\mu_g},$$

où $f_1, \ldots, f_k$ sont des fonctions test de supports disjoints. Nous utilisons l'estimation donnée par la propriété $(II)$ du théorème 9 pour des capacités. Puis, un bon choix pour la mesure de $\mu$ (voir théorème 9) donne les bornes supérieures désirées. Dans le cas du laplacian de Bakry–Émery $\Delta_\phi = \Delta_g + \nabla_g\phi\nabla_g$, nous utilisons aussi une caractérisation variationelle ; cependant il faut faire attention au choix de $r_0$ dans le théorème 9 : on doit tenir compte de $\phi$. En gros, le choix de $r_0$ est effectué via le théorème de comparaison des volumes donné par Wei et Wylie [40] pour des variétés Bakry–Émery.
Pour les valeurs propres du laplacien de Bakry–Émery , les bornes supérieures que nous obtenons mènent à une inégalité de type de Buser en termes de la borne inférieure de la courbure de Bakry–Émery–Ricci et des bornes de la fonction pondérée. Cependant, il existe une preuve simple et directe pour l'inégalité de type de Buser utilisant l'idée classique employée par Buser [6], Li et Yau [28]. Nous donnons cette preuve directe dans l'annexe.
Les méthodes employées dans le deuxième chapitre, pour étudier la relation entre géométrie extrinsèque et les valeurs propres du laplacien, sont différentes de la méthode ci-dessus. Nous utilisons deux idées principales : la première est l'idée de la construction CM qui permet de remplacer l'invariant $i(M)$ par des versions modifiées. La deuxième est l'inégalité universelle pour les valeurs propres obtenues par El Soufi, Harrell et Ilias [17] qui peut s'appliquer dans le cas des sous-variétés de l'espace projectif complexe.

# Introduction

## Motivations and historical backgrounds

Spectral geometry investigate the relationships between eigenvalues of natural operators like the Laplace–Beltrami operator, Schrödinger operator, etc. and other geometric invariants. The purpose of this thesis is to study these relationships through finding upper bounds for eigenvalues of the natural operators in terms of geometric data.
Let $(M, g)$ be a compact orientable $m$-dimensional Riemannian manifold. The Laplace–Beltrami operator $\Delta_g$ is defined as $\Delta_g = -\mathrm{div}\nabla_g$. It is well known that the spectrum of the Laplace–Beltrami operator acting on functions is discrete and consists of a non-decreasing sequence $\{\lambda_k(M, g)\}_{k=1}^{\infty}$ of eigenvalues each occurring with finite multiplicity. The Weyl law gives the asymptotic behavior of the eigenvalues of the Laplace–Beltrami operator :

$$\lambda_k(M, g)\left(\frac{\mu_g(M)}{k}\right)^{\frac{2}{m}} \sim \alpha_m, \quad k \to \infty$$

where $\mu_g$ is the Riemannian measure associated with $g$, $\alpha_m = 4\pi^2\omega_m^{-\frac{2}{m}}$ and $\omega_m$ is the volume of the unit ball in the standard $\mathbb{R}^m$.
One of the central questions that we study is to find upper bounds for the eigenvalues of the Laplace–Beltrami operator which are consistent with the Weyl law in the following sense :
For every $m$-dimensional compact Riemannian manifold $(M, g)$, we want to obtain a quantity $C$ such that for every $k \in \mathbb{N}^*$ we have

$$\lambda_k(M, g)\left(\frac{\mu_g(M)}{k}\right)^{\frac{2}{m}} \leq C. \tag{1}$$

First we may ask if one can find such a constant depending only on the dimension $m$. It turns out that the normalized first non-zero eigenvalue, $\lambda_2(M, g)\mu_g(M)^{\frac{2}{m}}$, of the Laplacian on compact manifolds cannot be bounded from above in terms of the dimension (see for example [11], [13], [30], [39]). Therefore, upper bounds should depend on some geometric data. In order to be consistent with the Weyl law asymptotically, we look for upper bounds which depend on the dimension and other geometric data such that the leading term depends only on the dimension as $k$ tends to the infinity. A typical example of such upper bounds given by the Buser inequality [6] (see also [16] and [28]) :
There exists a constant $\alpha_m$ depending only on $m$ such that for every $m$-dimensional compact Riemannian manifold $(M, g)$ with $\mathrm{Ricci}_g(M) \geq -\kappa^2(m-1)$ for some $\kappa \in \mathbb{R}$ and for

every $k \in \mathbb{N}^*$ :

$$\lambda_k(M,g)\left(\frac{\mu_g(M)}{k}\right)^{2/m} \leq \frac{(m-1)^2}{4}\kappa^2\left(\frac{\mu_g(M)}{k}\right)^{2/m} + \alpha_m. \tag{2}$$

One can see that this upper bound is consistent with the Weyl law, since the right-hand side of inequality asymptotically depends only on the dimension and on no geometric data. Finding geometric data that controls eigenvalues was investigated using different approaches. Other examples are about conformal upper bounds for the eigenvalues. In the particular case of the first positive eigenvalue[5] for compact Riemannian manifolds, El Soufi and Ilias [18] (see also [20]) showed that an inequality similar to Inequality (1) for the first positive eigenvalue $\lambda_2$ of the Laplacian holds with a constant $C_m([g])$ which depends on the conformal class $[g]$ of the metric $g$ (namely, the conformal volume introduced by Li and Yau [29] who proved the same result but in dimension 2). Yang and Yau [43] (see also [29]) proved Inequality (1) for $\lambda_2$ with a constant depending only on the genus of the surface. In 1993, Korevaar [27] generalized these results to higher order eigenvalues. In particular, for Riemannian surfaces, he gave an affirmative answer to Yau's conjecture [44]. More precisely, Korevaar obtained the following conformal upper bounds :
(i) If $(M^m, g)$ is a compact Riemannian manifold of dimension $m$, then for every $k \in \mathbb{N}^*$,

$$\lambda_k(M,g)\left(\frac{\mu_g(M)}{k}\right)^{\frac{2}{m}} \leq c_m([g]), \tag{3}$$

where $c_m([g])$ is a constant depending only on the conformal class $[g]$ of the metric $g$.
(ii) If $(\Sigma_\gamma, g)$ is a compact orientable surface of genus $\gamma$, then for every $k \in \mathbb{N}^*$,

$$\lambda_k(\Sigma_\gamma, g)\frac{\mu_g(\Sigma_\gamma)}{k} \leq C(\gamma+1), \tag{4}$$

where $C$ is a universal constant. Note that having only topological invariants as upper bounds in higher dimensions is hopeless because of the result of Colbois and Dodziuk [11]. However, Inequalities (3) and (4) are not consistent asymptotically with the Weyl law in the sense that we present in the beginning. Indeed, the constants in the right-hand side of the inequalities still depend either on the conformal class of the metric or on the topological invariant, genus, as $k$ tends to the infinity. Now, two natural questions arise :

**Question 1.** *Can one give a more explicit description for the conformal constant in Inequality (3) ?*

**Question 2.** *Can one obtain conformal upper bounds depending asymptotically only on the dimension and not on the geometry ?*

Answering these two questions is the main goal of the first chapter.

Another aspect of spectral geometry is to investigate of the relation between the extrinsic geometry of submanifolds and the spectrum of the Laplacian. One of the well-known

[5] Note that the first eigenvalue $\lambda_1$ of the Laplacian for compact Riemannian manifolds is zero.

extrinsic invariants is the mean curvature vector field of a submanifold. In this regard, we can mention the Reilly inequality [34] for immersed $m$-dimensional submanifolds of $\mathbb{R}^N$

$$\lambda_2(M) \leq \frac{m}{\mathrm{Vol}(M)} \|H(M)\|_2^2,$$

where $\|H(M)\|_2$ is the $L^2$-norm of the mean curvature vector field of $M$. For higher eigenvalues, it follows from results of El Soufi, Harrell and Ilias [17] and the recursion formula of Cheng and Yang [8, Corollary 2.1] that for every $k \in \mathbb{N}^*$,

$$\lambda_k(M) \leq R(m)\|H(M)\|_\infty^2 \, k^{2/m},$$

where $\|H(M)\|_\infty$ is the $L^\infty$-norm of $H(M)$ and $R(m)$ is a constant depending only on $m$. We are interested in extrinsic invariants which do not depend on the derivatives of the metric, excluding for instance curvature.
In [12], Colbois, Dryden and El Soufi studied the relation between the eigenvalues of the Laplace–Beltrami operator and an extrinsic invariant of submanifolds of $\mathbb{R}^N$, called the *intersection index* $i(M)$ defined as follows : For a compact $m$-dimensional immersed submanifold $M$ of $\mathbb{R}^{m+p}$, $p > 0$, the *intersection index* is given by

$$i(M) = \sup_{\Pi} \sharp(M \cap \Pi),$$

where $\Pi$ runs over the set of all $p$-planes that are transverse to $M$ ; if $M$ is not embedded, we count multiple points of $M$ according to their multiplicities. It turns out that Inequality (1) holds for eigenvalues of the Laplacian acting on a submanifold of $\mathbb{R}^N$ with a constant $C$ depending only on the intersection index $i(M)$ and the dimension of the submanifold.

$$\lambda_k(M) \left(\frac{\mathrm{Vol}(M)}{k}\right)^{2/m} \leq c(m) i(M)^{2/m}.$$

A remarkable consequence of this result concerns algebraic submanifolds. It gives an upper bound depending only on the "degree" for eigenvalues of the Laplacian on compact real algebraic manifolds. Notice that these results are not stable under "*small*" perturbations, since the intersection index might dramatically change. One can now ask the following questions :

**Question 3.** *Can one replace the "intersection index" by a modified version so that it would be stable under "small perturbations" ?*

**Question 4.** *Can one obtain an algebraic invariant like the degree as an upper bound for eigenvalues of the Laplacian acting on complex submanifolds of the complex projective space ?*

Our focus in the second chapter is to answer these questions. We extend the work of Colbois, Dryden and El Soufi in two directions. The first one consists in replacing the intersection index $i(M)$ by invariants of the same nature which are stable under "small" perturbations. The second direction concerns complex submanifolds of the complex projective space $\mathbb{C}P^N$.

Finally, we consider Schrödinger operators on a compact Riemannian manifold $(M, g)$. The eigenvalues of the Schrödinger operator $L = \Delta_g + q$, where $q$ is a continuous function on

$M$, constitute a non-decreasing unbounded sequence of real numbers. Using the variational characterization, one can see that the first eigenvalue is controlled by the mean value of the potential $q$. For the second eigenvalue $\lambda_2(L)$, an upper bound in terms of the mean value of the potential $q$ and a conformal invariant was obtained by El Soufi and Ilias [18, Theorem 2.2] :

$$\lambda_2(\Delta_g + q) \leq m \left( \frac{V_c([g])}{\mu_g(M)} \right)^{\frac{2}{m}} + \frac{\int_M q d\mu_g}{\mu_g(M)},$$

where $V_c([g])$ is the conformal volume that is defined by Li and Yau [29] which only depends on the conformal class $[g]$ of the metric $g$.
For a Riemannian surface $(\Sigma_\gamma, g)$ of genus $\gamma$, one replace the conformal volume by the genus $\gamma$ of the surface. Now the following interesting question arises.

**Question 5.** *Can one control the eigenvalues of the Schrödinger operator $L$ in terms of the mean value of the potential $q$ and geometric invariants of $M$ ?*

This question was investigated by Grigori'yan, Netrusov and Yau [24]. They gave an affirmative answer to the question when the Schrödinger operator $L$ is positive (see (3.5)). In general, they obtained upper bounds in terms of integral quantities depending on the potential $q$ (see inequalities (3.4), (3.5)) imposing some conditions on the metric ; however, their results are not consistent with the Weyl law regarding the power of $k$, except in dimension 2. In dimension 2, the upper bound they obtained [24, Theorem 5.4] depends on the genus and on integral quantities depending on the potential $q$.

In the third chapter, we obtain upper bounds which generalize and improve the results of [24] without imposing any geometric constraint, and which are asymptotically consistent with the Weyl law.

As a further application, we get upper bounds for the eigenvalues of weighted Laplace operators also called Bakry–Émery Laplace operators. One can ask the following question

**Question 6.** *What is the interplay between the eigenvalues of the Bakry–Émery Laplacian and geometric invariants of the weighted measure ?*

The last section of the third chapter is devoted to study the eigenvalues of the Bakry–Émery Laplacian and deals with Question 6.

## Statement of results

We are interested in the compact and orientable case and henceforth will always assume our manifolds to be compact and orientable. In what follows we present the main theorems of the thesis which also give answers to the questions asked in the previous section.

**Chapter 1.** In the same vein as the results of Korevaar, our aim is to obtain conformal upper bounds which are also consistent with the Weyl law. The main feature of our approach is that the modification we propose consists in taking a constant $C$ in Inequality (1) as the sum of two quantities : one of them depends on the conformal class $[g]$ or the genus $\gamma$ for Riemannian surfaces and tends to zero as $k$ tends to infinity, and the second

one depends only on the dimension. In order to state our results we need to introduce the following conformal invariant.
Let $(M, g)$ be a compact Riemannian manifold of dimension $m$, we define its *min-conformal* volume as follows :

$$V([g]) = \inf\{\mu_{g_0}(M) \ : \ g_0 \in [g], \ \mathrm{Ricci}_{g_0} \geq -(m-1)\}.$$

The following theorem gives an affirmative answer to both Questions 1 and 2.

**Theorem 1.** *For each integer $m \geq 2$, there exist two constants $A_m$ and $B_m$ such that, for every Riemannian manifold $(M, g)$ of dimension $m$ and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(M, g)\left(\frac{\mu_g(M)}{k}\right)^{\frac{2}{m}} \leq A_m\left(\frac{V([g])}{k}\right)^{\frac{2}{m}} + B_m.$$

*In particular in dimension two, there exist absolute constants $A$ and $B$ so that for every Riemannian surface $\Sigma_\gamma$ of genus $\gamma$ and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Sigma_\gamma, g)\frac{\mu_g(\Sigma_\gamma)}{k} \leq A\frac{\gamma}{k} + B. \tag{5}$$

Inequality (5) gives an upper bound to the topological spectrum introduced by Colbois and El Soufi [13] and can be compared with the lower bound they obtained [13, page 341].

The principal advantage of our approach lies in the fact that it enables us to recover Inequality (1) with a constant depending only on the dimension, for any integer $k$ that exceeds a threshold depending only on $[g]$ or $\gamma$. The following inequalities are direct consequences of Theorem 1 :
There exist a constant $B' > 0$ and, for each $m \geq 2$, a constant $B'_m > 0$ such that the following properties hold :
(1) For any compact Riemannian manifold $(M, g)$ of dimension $m \geq 2$, there exists an integer $k_0([g])$ depending only on the conformal class of $g$, such that, for every $k \geq k_0([g])$,

$$\lambda_k(M, g)\left(\frac{\mu_g(M)}{k}\right)^{\frac{2}{m}} \leq B'_m;$$

(2) For any compact Riemannian surface $(\Sigma_\gamma, g)$ of genus $\gamma$, there exists an integer $k_0(\gamma)$ depending only on $\gamma$, such that, for every $k \geq k_0(\gamma)$,

$$\lambda_k(\Sigma_\gamma, g)\frac{\mu_g(\Sigma_\gamma)}{k} \leq B'.$$

In the last section of the first chapter we study the Steklov eigenvalue problem as another illustration of our method. Recently, Girouard and Polterovich [22, Theorem 1.2] (see also [14], [19] and [21]) proved the following inequality for the Steklov eigenvalues of a compact Riemannian surface $(\Sigma_\gamma, g)$ of genus $\gamma$ and $\kappa$ boundary components :

$$\sigma_k(\Sigma_\gamma)\ell_g(\partial\Sigma_\gamma) \leq 2\pi(\gamma + \kappa)k,$$

where $\ell_g(\partial\Sigma)$ is the length of the boundary. For compact Riemannian surfaces, we obtain

**Theorem 2.** *Let $(\Sigma_\gamma, g)$ be a compact oriented Riemannian surface with genus $\gamma$, and $\Omega$ be a subdomain of $\Sigma_\gamma$. Then*

$$\sigma_k(\Omega)\ell_g(\partial\Omega) \le A\gamma + Bk, \tag{6}$$

*where $A$ and $B$ are universal constants.*

We should notice that our constants in spite of the results of [19], [21] and [22] are far from being sharp. In fact, our method can not lead to sharp constants. In higher dimension, we obtain an upper bound which depends on the conformal class of the metric and the isoperimetric ratio (see Theorem 1.4.1). One can compare it with results of Colbois, El Soufi and Girouard [14].

**Chapter 2.** In this chapter we extend the results of [12] as answers to Questions 3 and 4. We define the modified version of the intersection index $i(M)$ studied in [12] as follows. Let $G$ be the Grassmannian of all $m$-vector spaces in $\mathbb{R}^{m+p}$ endowed with an invariant Haar measure with total measure 1. Let $0 < \varepsilon < 1$ and $D$ be any open subdomain of $M$ such that $M \setminus D$ is a smooth manifold with the smooth boundary and $\mathrm{Vol}(D) \le \varepsilon \mathrm{Vol}(M)$. We denote $M \setminus D$ by $M_\varepsilon^D$ and $\sup_{P \perp H} \sharp(M_\varepsilon^D \cap P)$ by $i_H(M_\varepsilon^D)$, where $P$ is an affine $p$-plane orthogonal to $H$. The *$\varepsilon$-mean intersection index* is defined as :

$$\bar{\imath}^\varepsilon(M) := \inf_D \int_G i_H(M_\varepsilon^D)dH,$$

where $D$ runs over all regions whose measure is smaller than $\varepsilon\mathrm{Vol}(M)$ and $M \setminus D$ is a smooth manifold with smooth boundary.
Similarly, for $r > 0$, we define the *$(r, \varepsilon)$-local intersection index* as :

$$\bar{\imath}_r^\varepsilon(M) := \inf_D \sup_{x \in M_\varepsilon^D} \int_G i_H(M_\varepsilon^D \cap B(x,r))dH,$$

where $B(x, r) \subset \mathbb{R}^{m+p}$ is an Euclidean ball centered at $x$ and of radius $r$ and $D$ runs over all regions whose measure is smaller than $\varepsilon\mathrm{Vol}(M)$ and $M \setminus D$ is a smooth manifold with smooth boundary.
It is worth noting that these new notions are stable under *$\varepsilon$-small perturbations*[6]. We obtain the following theorem

**Theorem 3.** *There exist positive constants $c_m, \alpha_m$ and $\beta_m$ depending only on $m$ such that for every compact $m$-dimensional immersed submanifold $M$ of $\mathbb{R}^{m+p}$ and every $k \in \mathbb{N}^*$ and $\varepsilon > 0$, we have*

$$\lambda_k(M)\, Vol(M)^{2/m} \le c_m \frac{\bar{\imath}^\varepsilon(M)^{2/m}}{(1-\varepsilon)^{1+2/m}} k^{2/m},$$

$$\lambda_k(M) \le \alpha_m \frac{1}{(1-\varepsilon)r^2} + \beta_m \frac{\bar{\imath}_r^\varepsilon(M)^{2/m}}{(1-\varepsilon)^{1+2/m}} \left(\frac{k}{Vol(M)}\right)^{2/m}.$$

[6] Let $\varepsilon$ be a positive number, $\varepsilon < 1$. By "$\varepsilon$-small" perturbation we mean any perturbation in a region $D \subset M$ whose measure is at most equal to $\varepsilon$Vol(M).

In response to Question 4 we get

**Theorem 4.** *Let $M^m$ be an $m$-dimensional complex manifold admitting a holomorphic immersion $\phi : M \to \mathbb{C}P^N$. Then for every $k \in \mathbb{N}^*$ we have*

$$\lambda_{k+1}(M, \phi^* g_{FS}) \leq 2(m+1)(m+2)k^{\frac{1}{m}} - 2m(m+1),$$

*where $g_{FS}$ is the Fubini–Study metric on $\mathbb{C}P^N$.*

In particular, one has the above inequality for every complex submanifold of $\mathbb{C}P^N$. For a complex submanifold $M$ of $\mathbb{C}P^{m+p}$ of complex dimension $m$, we have

$$\mathrm{Vol(M)} = \deg(M)\mathrm{Vol}(\mathbb{C}P^m),$$

where $\deg(M)$ is the intersection number of $M$ with a projective $p$-plane in a generic position. Moreover, one can describe $M$ as the zero locus of a family of irreducible homogenous polynomials. The $\deg(M)$ is the product of degrees of the irreducible polynomials which describe $M$ (see for example [23, pages 171-172]). Therefore, we can rewrite Inequality (2.5) as

$$\lambda_{k+1}(M, \phi^* g_{FS})\mathrm{Vol}(M)^{\frac{1}{m}} \leq C(m) \deg(M)^{\frac{1}{m}} k^{\frac{1}{m}}. \tag{7}$$

Note that the power of $k$ is consistent with the Weyl law.

**Remark 1.** *for $k = 1$, one has*

$$\lambda_2(M, \phi^* g_{FS}) \leq 4(m+1),$$

*which is a sharp inequality since the equality holds for $\mathbb{C}P^m$. This sharp upper bound was also obtained by Bourguignon, Li and Yau [4, page 200] (see also [1]) under the assumption that the holomorphic immersion $\phi$ should be a full immersion. Theorem 2.1.2 gives us another proof of this sharp inequality without this assumption.*

**Chapter 3.** The first part of the third chapter concerns Question 5. We obtain the following extension of the results of [24] for the Schrödinger operator $L = \Delta_g + q$, $q \in C^0(M)$.

**Theorem 5.** *There exist constants $\alpha_m \in (0,1), B_m$ and $C_m$ depending only on $m$ such that for every $m$-dimensional compact Riemannian manifold $(M, g)$, every potential $q \in C^0(M)$ and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_g + q) \leq \frac{\alpha_m^{-1} \int_M q^+ d\mu_g - \alpha_m \int_M q^- d\mu_g}{\mu_g(M)} + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{2/m} + C_m \left(\frac{k}{\mu_g(M)}\right)^{2/m},$$

*where $V([g])$ is the min-conformal volume and $q^{\pm} = \max\{| \pm q|, 0\}$.*
*In particular, when the potential $q$ is positive one has*

$$\lambda_k(\Delta_g + q) \leq A_m \frac{1}{\mu_g(M)} \int_M q d\mu_g + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}, \tag{8}$$

*where $A_m = \alpha_m^{-1}$.*

Moreover, when the Schrödinger operator $L$ is positive we obtain

**Theorem 6.** *There exist constants $A_m > 1$, $B_m$ and $C_m$ depending only on $m$ such that if $L = \Delta_g + q$, $q \in C^0(M)$ is a positive operator, then for every compact $m$-dimensional Riemannian manifold $(M^m, g)$ and every $k \in \mathbb{N}^*$ we have*

$$\lambda_k(\Delta_g + q) \leq A_m \frac{\int_M q d\mu_g}{\mu_g(M)} + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}.$$

This leads to the following corollary for a Schrödinger operator $L = \Delta_g + q$, $q \in C^0(M)$.

**Corollary 1.** *There exist positive constants $A_m > 1$, $B_m$ and $C_m$ depending only on $m$ such that for every compact Riemannian manifold $(M, g)$, every potential $q \in C^0(M)$ and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_g + q) \leq A_m \frac{1}{\mu_g(M)} \int_M q d\mu_g + (1 - A_m)\lambda_1(\Delta_g + q) + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}.$$

For Riemannian surfaces, one can replace $V([g])$ by a topological invariant depending only on the genus. All of the above upper bounds are consistent with the Weyl law. As an interesting application of the above results, we find out the following upper bound for the eigenvalues of the weighted Laplacian or Bakry–Émery Laplacian $\Delta_\phi$ that we denote by $\lambda_k(\Delta_\phi)$. Given a Riemannian manifold $(M, g)$ and a function $\phi \in C^2(M)$, the corresponding weighted Laplace operator $\Delta_\phi$ is defined as follows :

$$\Delta_\phi = \Delta_g + \nabla_g \phi \cdot \nabla_g.$$

**Theorem 7.** *There exist constants $A_m > 1$, $B_m$ and $C_m$ depending on $m \in \mathbb{N}^*$, such that for every $m$-dimensional compact Riemannian manifold $(M, g)$, every $\phi \in C^2(M)$ and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_\phi) \leq A_m \frac{1}{\mu_g(M)} \|\nabla_g \phi\|^2_{L^2(M)} + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}.$$

There exist other results for complete Riemannian manifolds (see [33], [37], [41] and [42])) which present upper bounds for the first eigenvalue of the Bakry–Émery Laplace operator depending on the $L^\infty$-norm of $\nabla_g \phi$ and a lower bound of the Bakry–Émery Ricci tensor $\mathrm{Ricci}_\phi$. In that regard, the above theorem show that for compact manifolds, upper bounds for the first non-zero eigenvalue depends only on the $L^2$-norm of $\nabla_g \phi$ and on the conformal class of the metric. It also gives an upper bound for higher eigenvalues.
The interplay between the geometry of the manifold and the behavior of the weight function $\phi$ is mostly taken into account by means of a modified Ricci curvature called the Bakry–Émery Ricci curvature. Then, using techniques successfully applied in the case of the Laplace operator, we obtain the following results.

**Theorem 8.** *Let $(M, g, \phi)$ be a Bakry–Émery manifold with $\partial_r \phi \geq -\sigma$ for some $\sigma \geq 0$. Then, there exist constants $A(m)$ and $B(m)$ such that for every $k \in \mathbb{N}^*$,*

$$\lambda_k(\Delta_\phi) \leq A(m) \max\{\sigma^2, 1\} \left(\frac{V_\phi([g])}{\mu_\phi(M, g)}\right)^{2/m} + B(m) \left(\frac{k}{\mu_\phi(M)}\right)^{2/m},$$

*where the inequality $\partial_r\phi > -\sigma$ means that for every $x \in M$ the corresponding radial gradient of $\phi$ is at least equal to $-\sigma$.*

It is worth pointing out that in full generality, it is not possible to obtain upper bounds which do not depend on $\phi$ (see for instance [37, Section 2]). However, for compact manifolds with nonnegative Bary–Émery Ricci curvature we can find upper bounds which do not depend on $\phi$ :

**Corollary 2.** *Let $(M, g, \phi)$ be a Bakry–Émery manifold such that there exists $g_0 \in [g]$ with $\mathrm{Ricci}_\phi(M, g) \geq 0$. Then, there exists a positive constant $A(m)$ which depends only on the dimension such that for every $k \in \mathbb{N}^*$*

$$\lambda_k(\Delta_\phi) \leq A(m)\left(\frac{k}{\mu_\phi(M)}\right)^{2/m}.$$

## Methods

Here, we give a rough idea of the methods that we use to obtain our results. We begin by the main method that we introduce in this thesis. In the first chapter, we develop a new method to find upper bounds for eigenvalues of the Laplace–Beltrami operator. This method is based on extending the two elaborated constructions given by Grigor'yan, Netrusov and Yau [24], and Colbois and Maerten [16]. We call these two constructions the GNY-construction and the CM-construction, respectively.
Let us briefly recall the starting point to find upper bounds for the eigenvalues of the Laplacian. The classical min-max theorem for the eigenvalues of the Laplace–Beltrami operator implies that if we have a family $\{f_i\}_{i=1}^k$ of $k$ disjointly supported test functions then

$$\lambda_k(M, g) \leq \max_i \left\{\frac{\int_M |\nabla_g f_i|^2 d\mu_g}{\int_M f_i^2 d\mu_g}\right\}.$$

Hence, the problem of estimating eigenvalues primarily reduces to find disjoint capacitors[7] which shall be support of our test functions and secondarily to estimate $\frac{\int_M |\nabla_g f_i|^2 d\mu_g}{\int_M f_i^2 d\mu_g}$, $1 \leq i \leq k$, which is called Rayleigh quotient. The classical idea provide a family of balls as capacitors and plateau functions supported on each capacitor as test functions. This construction, which was investigated in [6], [10] and [28], works on compact manifolds and leads to results such as Inequality (2). Later on, Colbois and Maerten [16] introduced an elaborated construction which for example leads to have Inequality (2) for bounded subdomains of complete manifolds. We notice that in order to apply the method of Colbois and Maerten, we need to avoid a concentration of the metric i.e. small balls have to have small volumes. It follows that one can not get conformal upper bounds using only this construction. Another powerful construction, given by Grigor'yan, Netrusov and Yau [24] (comes from revising the original idea of Korevaar [27]), leads to have conformal upper bounds via capacities provided certain condition on manifolds. Our method consists in finding upper bounds via capacities without imposing extra condition on manifolds. It is

[7] Each pair $(F, G)$ of open sets in $M$ such that $F \subset G$ is called a *capacitor*.

mainly based on extending the GNY-construction and the CM-construction.
For each capacitor $(F,G)$ we define its capacity by

$$\operatorname{cap}_g(F,G)=\inf_{\varphi\in\mathcal{T}}\int_M|\nabla_g\varphi|^2d\mu_g,$$

where $\mathcal{T}=\mathcal{T}(F,G)$ is the set of all compactly supported Lipschitz functions on $M$ such that $\operatorname{supp}\varphi\subset \mathrm{G}^\circ=\mathrm{G}\setminus\partial\mathrm{G}$ and $\varphi\equiv 1$ in a neighborhood of $F$. If $\mathcal{T}(F,G)$ is empty, then $\operatorname{cap}_g(F,G)=+\infty$.
Let $\{(F_i,G_i)\}_{i=1}^k$ be a family of mutually disjoint capacitors with the following properties

- $\mu_g(F_i)\geq\alpha\,;$
- $\operatorname{cap}(F_i,G_i)\leq\beta.$

For every $\epsilon>0$, we construct our test functions $f_i\in\mathcal{T}(F_i,G_i)$, $1\leq i\leq k$, so that

$$\int_M|\nabla_g f|^2d\mu_g\leq\operatorname{cap}(F_i,G_i)+\epsilon.$$

For every $\epsilon>0$ and every $1\leq i\leq k$,

$$\frac{\int_M|\nabla_g f_i|^2d\mu_g}{\int_M f_i^2d\mu_g}\leq\frac{\beta+\epsilon}{\alpha}.$$

Hence,

$$\lambda_k\leq\frac{\beta}{\alpha}.$$

Assume that one has $F_1,\ldots,F_k$ so that $\mu_g(F_i)\geq\frac{\mu_g(M)}{Ck}$ for some constant $C>1$. Therefore, $\lambda_k\leq C\beta\frac{k}{\mu_g(M)}$. In the case where we look for conformal upper bounds, we estimate the capacity in terms of a conformally invariant. In dimension 2, the capacity is conformally invariant. In higher dimension $m>2$, one can estimate the capacity by the $m$-capacity[8] which is conformal invariant. By the Hölder inequality we get

$$\operatorname{cap}(F_i,G_i)\leq\operatorname{cap}_{[g]}^{(m)}(F_i,G_i)^{2/m}\left(\frac{\mu_g(M)}{k}\right)^{1-2/m}.$$

Therefore,

$$\lambda_k\left(\frac{\mu_g(M)}{k}\right)^{2/m}\leq C\operatorname{cap}_{[g]}^{(m)}(F_i,G_i)^{2/m}.$$

The GNY-construction gives a sufficient condition to control the $m$-capacity by a constant depending only on $m$. This condition can be reformulated as follows. If for a family $\mathcal{M}$ of

[8]Similarly, we can define the *m-capacity* as

$$\operatorname{cap}_{[g]}^{(m)}(F,G)=\inf_{\varphi\in\mathcal{T}}\int_M|\nabla_g\varphi|^m d\mu_g.$$

compact manifolds of dimension $m$, there is a constant depending only on $m$ such that for every $(M,g)\in\mathcal{M}$, some $g_0\in[g]$, every $x\in M$ and every $r>0$ we have

$$\mu_{g_0}(B(x,r))\leq B(m)r^m, \tag{9}$$

then we can obtain an upper bound depending only on $m$ for the $m$-capacities of the capacitors constructed by the GNY-construction. As a result we have Inequality (1) with a constant $C$ depending only on the dimension for this family of manifolds. Clearly we do not have this condition for the family of all compact manifolds. For instance, family of Reimannian manifolds with non-negative Ricci curvature satisfy this condition, but family of Riemannian manifolds with negative Ricci curvature do not meet this condition. However, we always have "locally" such a property for family of all compact manifolds.
The idea of our construction is as follows. We start with the GNY-construction which proceeds by induction. As long as each of the capacitor stays in a ball of radius $r_0$, where $r_0$ is a given constant, we continue the process. If the number of capacitors built using this method is equal to $k$ then we obtain an upper bound for the $m$-capacity depending only on $m$. If not, we shall see that in the complement of the capacitors constructed by the GNY-construction, one can apply the CM-construction and get a conformal upper bound for the $m$-capacity. We notice that the CM-construction in itself does not lead to have conformal upper bounds; however, with this approach to combining two constructions in an appropriate way, we get an estimate for the $m$-capacity as follows :

$$\mathrm{cap}^{(m)}_{[g]}(F_i,G_i)\leq\beta_1+\beta_2,$$

where $\beta_1$ is a constant depending only on $m$ and $\beta_2$ is a conformal invariant which tends to zero when $k$ goes to infinity. We now summarize our construction in the following theorem.

**Theorem 9.** *Let $(M^m,g,\mu)$ be a compact Riemannian manifold with a non-atomic Borel measure*[9] *$\mu$. Then there exist positive constants $c(m)\in(0,1)$ and $\alpha(m)$ depending only on the dimension such that for every $k\in\mathbb{N}^*$ there exists a family $\{(F_i,G_i)\}_{i=1}^k$ of capacitors with the following properties :*

*(I)* $\mu(F_i)>c(m)\frac{\mu(M)}{k}$,

*(II)* $\mathrm{cap}_g(F_i,G_i)\leq\frac{\mu_g(M)}{k}\left[\frac{1}{r_0^2}\left(\frac{V([g])}{\mu_g(M)}\right)^{2/m}+\alpha(m)\left(\frac{k}{\mu_g(M)}\right)^{2/m}\right]$.

*where* $r_0=\frac{1}{1600}$.

**Remark 2.** *We can partly state the above theorem in a general setting of metric measure spaces (see Theorem 1.2.1).*

In the third chapter, we use a modification of the above method to estimate the eigenvalues of Schrödinger operators and Bakry–Émery Laplace operators. When we consider the Schrödinger operator $L=\Delta_g+q$, having the following inequality as a result of the variational characterization,

$$\lambda_k(\Delta_g+q)\leq\max_{i\in\{1,\dots,k\}}\frac{\int_M|\nabla_g f_i|^2d\mu_g+\int_M f_i^2qd\mu_g}{\int_M f_i^2d\mu_g},$$

[9] A measure $\mu$ on a metric space $(X,d)$ is called non-atomic if and only if $\mu(\{x\})=0$ for every $x\in X$.

where $f_1, \ldots, f_k$ are disjointly supported test functions, allows us to use the estimate given in the property $(II)$ of Theorem 9 for capacities. Finally, making right choices for the measure $\mu$ in Theorem 9 leads to get desired upper bounds. For the Bakry–Émery Laplace operator $\Delta_\phi = \Delta_g + \nabla\phi\nabla$, we also have variational characterization for its eigenvalues ; however we have to be careful about the right choice of $r_0$ in Theorem 9 in order to apply it, where the properties of $\phi$ comes into play. Roughly speaking, the choice of $r_0$ somehow is affected by the volume comparison theorem proven by Wei and Wylie [40] for Bakry–Émery manifolds which is the key fact.
Moreover, for the eigenvalues of the Bakry–Émery Laplace operator, the conformal upper bound we obtain leads also to a Buser type upper bound for eigenvalues in terms of the lower bound for the Bakry–Émery Ricci curvature and the bound of the weighted function. However, there is a direct and simple proof for the Buser type upper bound using the classic idea used by Buser [6], Li and Yau [28]. We give this direct proof in the appendix.
The method we use in the second chapter to study the relation between extrinsic geometry and the eigenvalues of the Laplacian is different from the above method. The first key tool is the CM-construction. It allows us to replace the intersection index $i(M)$ by modified versions. The second key tool is the universal inequality for eigenvalues proved by El Soufi, Harrell and Ilias [17] that can be applied when we consider eigenvalues of the Laplacian on submanifolds of the complex projective space.

# Chapitre 1

# Bornes supérieures conformes pour les valeurs propres du laplacien et du problème de Steklov

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# Conformal Upper bounds for the eigenvalues of the Laplacian and Steklov problem

## 1.1 Introduction

Let $(M,g)$ be a compact orientable $m$-dimensional Riemannian manifold. It is well known that the spectrum of the Laplace operator acting on functions is discrete and consists of a nondecreasing sequence $\{\lambda_k(M,g)\}_{k=1}^{\infty}$ of eigenvalues each occurring with finite multiplicity. If $M$ has a smooth boundary then the same conclusion is valid for Dirichlet, Neumann or other reasonable boundary conditions. By the Weyl law, the asymptotic behavior of $\lambda_k$ is given by (see e.g. [3])

$$\lambda_k(M,g) \sim \alpha_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}, \quad k \to \infty \tag{1.1}$$

where $\mu_g$ is the Riemannian measure associated with $g$, $\alpha_m = 4\pi^2 \omega_m^{-\frac{2}{m}}$ and $\omega_m$ is the volume of the unit ball in the standard $\mathbb{R}^m$.
A natural question suggested by this asymptotic formula is the following

**Question.** *Does there exists a constant $C_m$ depending only on the dimension $m$ such that we have*

$$\lambda_k(M,g)\mu_g(M)^{\frac{2}{m}} \le C_m k^{\frac{2}{m}} \tag{1.2}$$

*for every $k \in \mathbb{N}^*$?*

An abundant literature has been devoted to this issue starting with Urakawa's paper [39]. It turns out that $\lambda_2(M,g)\mu_g(M)^{\frac{2}{m}}$ cannot be bounded above only in terms of $m$ (see for example [5], [11], [13], [30]). Consequently, the answer to Question 1 is negative.

In the particular case of the first positive eigenvalue, El Soufi and Ilias [18] (see also [20]) showed that an inequality like (1.2) holds with a constant $C_m([g])$ that depends on the conformal class $[g]$ of the metric $g$ (namely, the conformal volume introduced by Li and Yau [29] who proved the same but in dimension 2). In the case of surfaces, Yang and Yau [43] proved Inequality (1.2) with a constant that only depends on the genus of the surface. In 1993, Korevaar [27] generalized these results to higher order eigenvalues. More precisely, Korevaar obtained the following upper bounds:
(i) If $(M^m,g)$ is a compact Riemannian manifold of dimension $m$, then for every $k \in \mathbb{N}^*$,

$$\lambda_k(M,g)\mu_g(M)^{\frac{2}{m}} \le c_m([g])k^{\frac{2}{m}}, \tag{1.3}$$

where $c_m([g])$ is a constant depending only on the conformal class $[g]$ of the metric $g$.
(ii) If $(\Sigma_\gamma, g)$ is a compact orientable surface of genus $\gamma$, then for every $k \in \mathbb{N}^*$,

$$\lambda_k(\Sigma_\gamma, g)\mu_g(\Sigma_\gamma) \leq C(\gamma+1)k, \tag{1.4}$$

where $C$ is a universal constant.

Notice that Inequality (1.4) provides an affirmative answer to Yau's conjecture [44, page 19]. Korevaar's results have been discussed by Gromov [26] and revisited by Grigor'yan and Yau [25] and Grigor'yan, Netrusov and Yau [24] who proposed different proofs.

Another important result in this direction was obtained by Buser [6] who proved that if $(M^m, g)$ is a compact $m$-dimensional Riemannian manifold whose Ricci curvature satisfies $\mathrm{Ricci}_g \geq -(m-1)a^2$, then for every $k \in \mathbb{N}^*$,

$$\lambda_k(M, g) \leq \frac{(m-1)^2}{4}a^2 + \beta_m \left(\frac{k}{\mu_g(M)}\right)^{2/m}, \tag{1.5}$$

where $\beta_m$ is a constant depending only on $m$.
Colbois and Maerten ([16] Thm 1.3) proved a similar result for bounded domains in a complete manifold under Neumann boundary conditions.

In the same vein as the results of Korevaar and Buser, our aim in the present work is to understand how Inequality (1.2) can be modified into a valid one. We obtain results that generalize those of Korevaar, Buser, and Colbois and Maerten mentioned above. The main feature of our approach is that the modification we propose consists in adding a term (depending on the conformal class $[g]$ or the genus $\gamma$) to the right hand side of (1.2), instead of letting the constant $C_m$ depend on $[g]$ or $\gamma$ as in Korevaar's Inequalities (1.3) and (1.4). The principal advantage of our approach lies in the fact that it enables us to recover the inequality (1.2) for any integer $k$ that exceeds a threshold depending only on $[g]$ or $\gamma$ (see Corollary 1.1.3 below).

In order to state our main result we need to introduce the following conformal invariant. If $(M, g)$ is a compact Riemannian manifold of dimension $m$, we define its **min-conformal** volume as follows:

$$V([g]) = \inf\{\mu_{g_0}(M) \ : \ g_0 \in [g], \ \mathrm{Ricci}_{g_0} \geq -(m-1)\}.$$

Denoting by $\rho^-(g)$ the smallest number $a \geq 0$ such that $\mathrm{Ricci}_g \geq -(m-1)a^2$, one can easily check that

$$\begin{aligned} V([g]) &= \inf\{\mu_{g'}(M)\rho^-(g')^{\frac{m}{2}} \ : \ g' \in [g]\} \\ &= \inf\{\rho^-(g')^{\frac{m}{2}} \ : \ g' \in [g]\,,\, \mu_{g'}(M) = 1\}. \end{aligned} \tag{1.6}$$

**Theorem 1.1.1.** *There exist, for each integer $m \geq 2$, two constants $A_m$ and $B_m$ such that, for every compact Riemannian manifold $(M, g)$ of dimension $m$ and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(M, g)\mu_g(M)^{\frac{2}{m}} \leq A_m V([g])^{\frac{2}{m}} + B_m k^{\frac{2}{m}}. \tag{1.7}$$

It is important to notice that the constant $B_m$ in Inequality (1.7) cannot be equal to the constant $\alpha_m$ in the Weyl law. Indeed, it follows from [13, Corollary 1] that such a $B_m$ must satisfy : $B_m \geq m\omega_m^{\frac{2}{m}}$. On the other hand, Inequality (1.7) also gives an upper bound on the conformal spectrum introduced by Colbois and El Soufi [13] and shows that its asymptotic behavior obeys a Weyl type law.

Now, if a metric $g$ is conformally equivalent to a metric $g_0$ with $\mathrm{Ricci}_{g_0} \geq 0$, then $V([g]) = 0$ (see equality (1.6)). This leads to the following

**Corollary 1.1.1.** *(see [27]) If a compact Riemannian manifold $(M, g)$ of dimension $m \geq 2$ is conformally equivalent to a Riemannian manifold with nonnegative Ricci curvature, then*

$$\lambda_k(M, g)\mu_g(M)^{\frac{2}{m}} \leq B_m k^{\frac{2}{m}}, \tag{1.8}$$

*where $B_m$ is a constant depending only on $m$.*

In the case of a compact orientable surface $\Sigma_\gamma$ of genus $\gamma$, the uniformization theorem tells us that any Riemannian metric $g$ on $\Sigma_\gamma$ is conformally equivalent to a metric of constant curvature. If $\gamma \geq 2$, then $g$ is conformally equivalent to a hyperbolic metric $g_\gamma$. Thus, $V([g]) \leq \mu_{g_\gamma}(\Sigma_\gamma) = 4\pi(\gamma - 1)$, where the last equality follows from Gauss-Bonnet Theorem. If $\gamma = 0, 1$, then $g$ is conformally equivalent to a positive constant curvature metric or a flat metric, respectively. Thus, $V([g]) = 0$ in the last two cases. Substituting in (1.7), one obtains the following improvement of Korevaar's Inequality (1.4).

**Corollary 1.1.2.** *There exist two constants $A$ and $B$ such that, for every compact Riemannian surface $(\Sigma_\gamma, g)$ of genus $\gamma$ and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Sigma_\gamma, g)\mu_g(\Sigma_\gamma) \leq A\gamma + Bk. \tag{1.9}$$

This result gives an upper bound to the topological spectrum introduced by Colbois and El Soufi [13] and can be compared with the lower bound they obtained [13, page 341].

In relation with Question 1, we have the following corollary which is a direct consequence of Inequalities (1.7) and (1.9).

**Corollary 1.1.3.** *There exist a constant $B' \in \mathbb{R}$ and, for each $m \geq 2$, a constant $B'_m \in \mathbb{R}$ such that the following properties hold.*
*(1) For any compact Riemannian manifold $(M, g)$ of dimension $m \geq 2$, there exists an integer $k_0([g])$ depending only on the conformal class of $g$, such that, for every $k \geq k_0([g])$,*

$$\lambda_k(M, g)\mu_g(M)^{\frac{2}{m}} \leq B'_m\, k^{\frac{2}{m}};$$

*(2) For any compact Riemannian surface $(\Sigma_\gamma, g)$ of genus $\gamma$, there exists an integer $k_0(\gamma)$ depending only on $\gamma$, such that, for every $k \geq k_0(\gamma)$,*

$$\lambda_k(\Sigma_\gamma, g)\mu_g(\Sigma_\gamma) \leq B'k.$$

For any relatively compact domain $\Omega$ with $C^1$ boundary in a Riemannian manifold $(M, g)$, we denote by $\{\lambda_k(\Omega, g)\}_{k\geq 1}$ the nondecreasing sequence of eigenvalues of the Neumann realization of the Laplacian in $\Omega$. The method we will use to prove Theorem 1.1.1 also allows us to obtain the following

**Theorem 1.1.2.** *Let $(M, g_0)$ be a complete Riemannian manifold of dimension $m \geq 2$ with $\mathrm{Ricci}_{g_0}(M) \geq -(m-1)$. Let $\Omega \subset M$ be a relatively compact domain with $C^1$ boundary and $g$ be any metric conformal to $g_0$. Then for every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Omega, g)\mu_g(\Omega)^{\frac{2}{m}} \leq A'_m \mu_{g_0}(\Omega)^{\frac{2}{m}} + B'_m k^{\frac{2}{m}}, \tag{1.10}$$

*where $A'_m$ and $B'_m$ are constants depending only on the dimension $m$.*

It is easy to see that we can derive from Theorem 1.1.1 and Theorem 1.1.2, inequalities of type (1.5) as obtained by Buser [6] and Colbois and Maerten [16] but with different constants.

The paper is organized as follows: In section 2 we introduce the main technical tool of the proof which consists in the construction of a suitable family of capacitors, using the methods of [GNY] and [CM]. The proofs of Theorem 1.1.1 and Theorem 1.1.2 are given in section 3. The last section is devoted to the Steklov eigenvalue problem. We prove that our method applies to this problem and give some upper bounds for the Steklov eigenvalues.

## 1.2 Construction of families of capacitors in an m-m Space

In this section, we present the main technical tool of this paper. Let us start by recalling some definitions. Throughout this section, the notation $(X, d, \mu)$ will designate a complete and locally compact metric-measure space ($m - m$ *space*) with a distance $d$ and a non-atomic finite Borel measure[1] $\mu$. We also assume that every ball in $X$ is pre-compact. Each pair $(F, G)$ of Borel sets in $X$ such that $F \subset G$ is called a *capacitor*.

**Definition 1.2.1.** *Given $\kappa > 1$ and $N \in \mathbb{N}^*$, we say that a metric space $(X, d)$ satisfies the $(\kappa, N)$-covering property if each ball of radius $r > 0$ can be covered by $N$ balls of radius $\frac{r}{\kappa}$.*

Similarly we define a local version of the covering property as follows:

**Definition 1.2.2.** *Given $\kappa > 1$, $\rho > 0$ and $N \in \mathbb{N}^*$, we say that a metric space $(X, d)$ satisfies the $(\kappa, N; \rho)$-covering property if each ball of radius $0 < r \leq \rho$ can be covered by $N$ balls of radius $\frac{r}{\kappa}$.*

**Lemma 1.2.1.** *If a metric space $(X, d)$ satisfies the $(\kappa, N; \rho)$-covering property (the $(\kappa, N)$-covering property), then for any $\lambda > 1$, it satisfies the $(\lambda, K; \rho)$-covering property (the $(\lambda, K)$-covering property) for some $K = K(\lambda, \kappa, N)$ that does not depend on $\rho$.*

The proof of the lemma when $(X, d)$ satisfies the $(\kappa, N)$-covering property is given in [24, Lemma 3.4]. For the $(\kappa, N; \rho)$-covering property, the same proof applies here verbatim.

**Definition 1.2.3.** *For any $x \in X$ and $0 \leq r \leq R$, we define the annulus $A(x, r, R)$ as*

$$A(x, r, R) := B(x, R) \setminus B(x, r) = \{y \in X : r \leq d(x, y) < R\}.$$

[1] A measure $\mu$ on a metric space $(X, d)$ is called non-atomic if and only if $\mu(\{x\}) = 0$ for every $x \in X$.

*For any annulus $A(x, r, R)$ and $\lambda \geq 1$, set $\lambda A := A(x, \lambda^{-1}r, \lambda R)$. Similarly, for any ball $B = B(x, r)$ we set $\lambda B := B(x, \lambda r)$. If $F \subseteq X$ and $r > 0$, we denote the $r$-neighborhood of $F$ by $F^r$, that is*

$$F^r = \{x \in X : d(x, F) \leq r\}.$$

In the following lemmas we recall two methods for metric construction of disjoint domains.

**Lemma 1.2.2.** *[24, Corollary 3.12] Let $(X, d, \mu)$ be an $m-m$ space satisfying the $(2, N)$-covering property. Then for every $n \in \mathbb{N}^*$, there exists a family $\mathcal{A} = \{(A_i, B_i)\}_{i=1}^n$ of capacitors in $X$ such that*

*(a) For each $i$, $A_i$ is an annulus and $\mu(A_i) \geq \frac{\mu(X)}{cn}$,*

*(b) $\{B_i\}_{i=1}^n$ are mutually disjoint and $B_i = 2A_i$,*

*where $c$ is a positive constant depending only on $N$ ( in fact one can take $c = 2 + 4K(1600, 2, N)$, where $K$ is the function defined in Lemma 1.2.1).*

**Lemma 1.2.3.** ([16, Corollary 2.3] and [15, Lemma 2.1]) *Let $(X, d, \mu)$ be an $m-m$ space satisfying the $(2, N; \rho)$-covering property. For every $n \in \mathbb{N}^*$, let $0 < r \leq \rho$ be such that for each $x \in X$, $\mu(B(x, r)) \leq \frac{\mu(X)}{4\tilde{N}^2 n}$, where $\tilde{N} = K(4, 2, N)$. Then there exists a family $\mathcal{A} = \{(A_i, A_i^r)\}_{i=1}^n$ of capacitors in $X$ such that*

*(a) for each $i$, $\mu(A_i) \geq \frac{\mu(X)}{2\tilde{N}n}$, and*

*(b) the subsets $\{A_i^r\}_{i=1}^n$ are mutually disjoint.*

In the original statement of Lemma 1.2.3, $(X, d)$ is supposed to have the $(4, \tilde{N}; \rho)$-covering property. According to Lemma 1.2.1, one can replace the $(4, \tilde{N}; \rho)$-covering property by the $(2, N; \rho)$-covering property. The main construction given in the following theorem results from a merging of the two previous lemmas. It consists in constructing a disjoint family of capacitors.

**Theorem 1.2.1.** *Let $(X, d, \mu)$ be an $m-m$ space with the non-atomic Borel measur $\mu$ satisfying the $(2, N; \rho)$-covering property. Then for every $n \in \mathbb{N}^*$, there exists a family of capacitors $\mathcal{A} = \{(F_i, G_i)\}_{i=1}^n$ with the following properties:*

*(i) $\mu(F_i) \geq \nu := \frac{\mu(X)}{8c^2 n}$, where $c$ is as in Lemma 1.2.2 ;*

*(ii) the $G_i$'s are mutually disjoint ;*

*(iii) the family $\mathcal{A}$ is such that either*

*(a) all the $F_i$'s are annuli and $G_i = 2F_i$, with outer radii smaller than $\rho$, or*

*(b) all the $F_i$'s are domains in $X$ and $G_i = F_i^{r_0}$, with $r_0 = \frac{\rho}{1600}$.*

*Proof of Theorem 1.2.1.* In order to find a desired family of capacitors, we start with the method used by Grigor'yan, Netrusov and Yau [24, proof of Theorem 3.5]. We will call their method *GNY-construction.* However we do not have the $(2, N)$-covering property in order to apply directly the GNY-construction. Roughly speaking, we will see that when an $m-m$ space $X$ has the local covering property (i.e. $(2, N; \rho)$-covering property), the GNY-construction is applicable to the "*massive part*" of X (i.e. where balls of radii $r_0$ have measure greater than $\nu$). If the number of capacitors built using the GNY-construction on the massive part is not equal to $n$, then we introduce a new measure on $X$. The support of this new measure is a subset of the complement of the massive part. We shall see that in this case the method of Colbois and Maerten (Lemma 1.2.3) that we will call *CM-construction*, is applicable.
Let us define

$$\tau_1 := \sup\{r : \mu(B(x,r)) \leq \nu \quad \forall x \in X\}.$$

If $\tau_1 \leq r_0$ then we follow the step 1 (see below). Otherwise we move on to the step 2 in order to apply the CM-construction.

**Step 1. Applying GNY-construction.** Assume $\tau_1 \leq r_0$.
We essentially follow the steps of the GNY-construction. However, it is necessary to make some adaptations since our covering property is of local nature. We use the same formalism and notations that is used in the GNY-construction (see [24, page 172]). Our goal is to construct by induction two sequences $\{\mathcal{A}_i\}$ and $\{\mathcal{B}_i\}$ where $\mathcal{A}_i$ is a family of annuli in $X$, and $\mathcal{B}_i$ is a family of balls that cover $\mathcal{A}_i$. These two families satisfy the following properties:

$(i)$ for each $a \in \mathcal{A}_i$ we have
$$\mu(a) \geq \nu \ ;$$

$(ii)$ the annuli $\{2a\}_{a\in\mathcal{A}_i}$ are disjoint ;

$(iii)$ for each $a \in \mathcal{A}_i$, the outer radius of $a$ is smaller than $\rho$ ;

$(iv)$ the following inclusions hold
$$\bigcup_{a\in\mathcal{A}_i} 2a \subset \bigcup_{b\in\mathcal{B}_i} \frac{1}{4}b \ ;$$

$(v)$ we have the inequality
$$\mu(\bigcup_{b\in\mathcal{B}_i} b) \leq c\nu i \ ;$$

$(vi)$ $|\mathcal{A}_1| = |\mathcal{B}_1| = 1$ and if $i > 1$ then

- either $|\mathcal{A}_i| = |\mathcal{A}_{i-1}| + 1$ and $|\mathcal{B}_i| \leq |\mathcal{B}_{i-1}| + 1$,
- or $|\mathcal{A}_i| = |\mathcal{A}_{i-1}|$ and $|\mathcal{B}_i| \leq |\mathcal{B}_{i-1}| - 1$,

where $|\mathcal{A}|$ denote the cardinal of the family $\mathcal{A}$ ;

$(vii)$ if $i > 1$, then $\mathcal{A}_{i-1} \subseteq \mathcal{A}_i$ ;

$(viii)$ if $i>1$, then $\bigcup_{b\in\mathcal{B}_{i-1}} b \subseteq \bigcup_{b\in\mathcal{B}_i} b$.

Observe that by $(vi)$ the sequence of $\{2|\mathcal{A}_i| - |\mathcal{B}_i|\}$ is strictly increasing with respect to $i$ and, since $2|\mathcal{A}_1| - |\mathcal{B}_1| = 1$, one has

$$2|\mathcal{A}_i| \geq i.$$

Notice that if we can continue the inductive process till $i = 2n$, then we get a family $\mathcal{A} = \mathcal{A}_{2n}$ of at least $n$ capacitors satisfying the desired properties $(i), (ii)$ and $(iii)(a)$ of Theorem 1.2.1. However here we only have a local covering property rather than a global one. In order to perform the induction, we will need to fix an upper bound on the radii of balls in $\mathcal{B}_i$ (this restriction is crucial to have property $(v)$). This restriction does not always allow us to continue the inductive process till $i = 2n$.

To start the induction, take $r \in (\tau_1, 2\tau_1]$. Then there exists a point $x_0 \in X$ such that

$$\mu(B(x_0, r)) \geq \nu.$$

We define $\mathcal{A}_1 = \{B(x_0, r)\}$ and $\mathcal{B}_1 = \{B(x_0, 8r)\}$. It is easy to see that properties $(i), (ii), (iii), (iv), (vi), (vii)$ and $(viii)$ are satisfied. Let us verify property $(v)$. Since $8r \leq 16\tau_1 < \rho$, by Lemma 1.2.1, one can cover $B(x_0, 8r)$ by $K(16, 2, N)$ balls of radii $r/2 < \tau_1$. Therefore,

$$\mu(B(x_0, 8r)) \leq K(16, 2, N)\nu < c\nu,$$

which proves the property $(v)$.
Assume now we have constructed $\mathcal{A}_1, \ldots, \mathcal{A}_i$ and $\mathcal{B}_1, \ldots, \mathcal{B}_i$ for some $i < 2n$. It follows from the property $(iv)$ for the family $\mathcal{B}_i$ that

$$\begin{aligned} \mu(X \setminus \bigcup_{b\in\mathcal{B}_i} b) &> \mu(X) - ic\nu > \mu(X) - 2nc\nu = \mu(X) - 2nc\frac{\mu(X)}{8c^2 n} \\ &= (1 - \frac{1}{4c})\mu(X) > \frac{\mu(X)}{2} > \nu, \end{aligned} \tag{1.11}$$

because $c > 1$. Hence, there exists $x_i \in X$ such that

$$\mu\big(B(x_i, r) \setminus \bigcup_{b\in\mathcal{B}_i} b\big) > \nu. \tag{1.12}$$

We define

$$\tau_{i+1} := \sup\{r : \mu\big(B(x, r) \setminus \bigcup_{b\in\mathcal{B}_i} b\big) \leq \nu \quad \forall x \in X\}.$$

At this stage the continuation of the construction process depends on the size of $\tau_{i+1}$.

- If $\tau_{i+1} > r_0$, we move on to the step 2.
- If $\tau_{i+1} \leq r_0$, we construct families $\mathcal{A}_{i+1}$ and $\mathcal{B}_{i+1}$ as follows.

We can assume that $r \in (\tau_{i+1}, 2\tau_{i+1}]$ in (1.12). We denote $\kappa$ the cardinal of:

$$B := \{b \in \mathcal{B}_i : \ B(x_i, 7 \times 4r) \cap \frac{1}{2}b \neq \emptyset\}.$$

Following the GNY-construction, we define $\mathcal{A}_{i+1}$ and $\mathcal{B}_{i+1}$ according to the following alternatives (for more details see [24, pages. 174–178]):

**Case** $\kappa = 0$: We define $\mathcal{A}_{i+1}$ and $\mathcal{B}_{i+1}$ by

$$\mathcal{A}_{i+1} = \mathcal{A}_i \cup \{B(x_i, r)\}, \quad \text{and} \quad \mathcal{B}_{i+1} = \mathcal{B}_i \cup \{B(x_i, 8r)\}.$$

**Case** $\kappa \geq 2$: We define $\mathcal{A}_{i+1}$ and $\mathcal{B}_{i+1}$ by

$$\mathcal{A}_{i+1} = \mathcal{A}_i, \quad \text{and} \quad \mathcal{B}_{i+1} = (\mathcal{B}_i \setminus \{\text{all balls in the set } B\}) \cup \{B(x_i, 98 \times 8r)\}.$$

Note that the ball $B(x_i, 98 \times 8r)$ contains all balls in $B$ (see [24, page 175]).

**Case** $\kappa = 1$: If there exists a ball $b = B(y, s) \in B$ such that

$$B(x_i, 2r) \cap \frac{1}{2}b \neq \emptyset,$$

then we define $\mathcal{A}_{i+1}$ and $\mathcal{B}_{i+1}$ by

$$\mathcal{A}_{i+1} = \mathcal{A}_i \cup \{A(y, \frac{1}{2}s, 8r)\} \quad \text{and} \quad \mathcal{B}_{i+1} = \mathcal{B}_i \cup \{B(x_i, 14 \times 8r)\}.$$

Notice that $A(y, \frac{1}{2}s, 8r) \subset B(x_i, 14 \times 8r)$ (see [24, page 177]).
Otherwise we define $\mathcal{A}_{i+1}$ and $\mathcal{B}_{i+1}$ like in the case $\kappa = 0$.

Now let us prove that these two families have the properties $(i) - (viii)$.
The properties $(vi), (vii)$ and $(viii)$ are clearly satisfied in each of the three cases. To check the conditions $(i)$, $(ii)$ and $(iv)$, we can use word-for-word the arguments given in [24, pages 173–178]. Indeed, this part of their proof is independent of covering properties.

Let us verify the condition $(v)$. In each of the three cases, we see that $|\mathcal{B}_{i+1} \setminus \mathcal{B}_i| = 1$. Let us denote by $b_{i+1}$ the unique ball in $\mathcal{B}_{i+1} \setminus \mathcal{B}_i$. According to the three cases, the radius $r_{i+1}$ of $b_{i+1}$ is at most $98 \times 8r$. Since $r \in (\tau_{i+1}, 2\tau_{i+1}]$, we have

$$r_{i+1} \leq 98 \times 8 \times 2\tau_{i+1} < 1600\tau_{i+1} \leq \rho, \tag{1.13}$$

where the last inequality follows from the assumption $\tau_{i+1} \leq r_0$. By Lemma 1.2.1, the ball $b_{i+1}$ can be covered by $K(1600, 2, N) < c$ balls with radii $\frac{r_{i+1}}{1600} \leq \tau_{i+1}$. Therefore

$$\begin{aligned} \mu(\bigcup_{b \in \mathcal{B}_{i+1}} b) &= \mu(\bigcup_{b \in \mathcal{B}_i} b) + \mu(b_{i+1} \setminus \bigcup_{b \in \mathcal{B}_i} b) \leq c\nu i + \mu(b_{i+1} \setminus \bigcup_{b \in \mathcal{B}_{i+1}} b) \\ &\leq c\nu i + K(1600, 2, N)\nu \leq c\nu i + c\nu \leq c\nu(i+1), \end{aligned}$$

which proves the condition $(v)$.
It remains to check the condition $(iii)$. For this, it is enough to verify that the outer radius of the annulus $a \in \mathcal{A}_{i+1} \setminus \mathcal{A}_i$ is smaller that $\rho$. One can see in each of the three cases, $\mathcal{A}_{i+1} \setminus \mathcal{A}_i \subset \mathcal{B}_{i+1} \setminus \mathcal{B}_i = \{b_{i+1}\}$. By Inequality (1.13), the radius of $b_{i+1}$ is smaller than $\rho$ and proves the condition $(iii)$ for $\mathcal{A}_{i+1}$.

**Step 2. Applying CM-construction.** Assume $\tau_i > r_0$ for some $1 \leq i \leq 2n$. It means that

- if $i = 1$, then $\mu(B(x, r_0)) \leq \nu$, for all $x \in X$ ;
- if $1 < i \leq 2n$, then $\mu(B(x, r_0) \setminus \bigcup_{b \in \mathcal{B}_{i-1}} b) \leq \nu$, for all $x \in X$.

We consider the $m - m$ space $(X, d, \tilde{\mu}_i)$ where

- $\tilde{\mu}_i := \mu$ if $i = 1$ ;
- $\tilde{\mu}_i(A) := \mu(A \setminus \bigcup_{b \in \mathcal{B}_{i-1}} b)$ if $1 < i \leq 2n$.

It follows from Inequality (1.11) and the above inequalities that

$$\tilde{\mu}_i(X) > \frac{\mu(X)}{2},$$

and

$$\tilde{\mu}_i(B(x, r_0)) \leq \frac{\mu(X)}{8c^2 n} \leq \frac{\tilde{\mu}_i(X)}{4\tilde{N}^2 n}.$$

Consequently, that the $m - m$ space $(X, d, \tilde{\mu}_i)$ satisfies the assumptions of Lemma 1.2.3. Therefore, there exists a family $\{(A_j, A_j^{r_0})\}$ of $n$ capacitors in $X$ such that the $A_j^{r_0}$'s are mutually disjoint and

$$\tilde{\mu}_i(A_j) \geq \frac{\tilde{\mu}_i(X)}{2\tilde{N}n} \geq \frac{\mu(X)}{8c^2 n}.$$

Since $\mu(A_j) \geq \tilde{\mu}_i(A_j)$, this family of capacitors satisfies the conditions $(i), (ii)$ and $(iii)(b)$ of Theorem 1.2.1. $\square$

The following proposition shows that for a sufficiently large integer $n$, it is always possible to apply the GNY-construction to obtain a family of $n$ capacitors satisfying the properties $(i), (ii)$ and $(iii)(a)$ of Theorem 1.2.1. The application of this observation to the eigenvalue problem is discussed in Remark 1.3.2 of the next section.

**Proposition 1.2.1.** *Let $(X, d, \mu)$ be a compact $m{-}m$ space satisfying the $(2, N; 1)$-covering property. Then there exists a positive integer $k_X$ such that for every $n > k_X$, there exists a family $\mathcal{A}$ of $n$ mutually disjoint capacitors in $X$ that satisfies the properties $(i), (ii)$ and $(iii)(a)$ of Theorem 1.2.1.*

*Proof.* Since $X$ is compact, we can cover $X$ by $T$ balls of radii $r_0 = \frac{1}{1600}$. Set

$$k_X = \frac{T}{4c^2}.$$

It is enough to show that for every $n > k_X$ and $1 \leq i \leq 2n$, we have $\tau_i \leq r_0$. Indeed, suppose that there exists an integer $j \leq 2n$ such that $\tau_j > r_0$. Then by the definition of $\tau_j$, we have the following inequality

$$\tilde{\mu}_j(B(x, r_0)) \leq \nu = \frac{\mu(X)}{8c^2 n}. \tag{1.14}$$

It follows from the above inequality that

$$\frac{\mu(X)}{2} \leq \tilde{\mu}_j(X) \leq \tilde{\mu}_j\left(\bigcup_{x_i \in X, 1 \leq i \leq T} B(x_i, r_0)\right) \leq T\frac{\mu(X)}{8c^2 n}.$$

Hence $n$ should be smaller than $\frac{T}{4c^2}$. Therefore, $\tau_j \leq r_0$ for every $j \leq 2n$. It follows that at the step $i = 2n$ of the inductive process (see the proof of Theorem 1.2.1 step 1), we have a family of $n$ mutually disjoint capacitors satisfying the proposition, which completes the proof. $\square$

## 1.3 Eigenvalues of the Laplace operator

In this section we apply Theorem 1.2.1 to a special case of $m - m$ spaces which are Riemannian manifolds, in order to prove Theorem 1.1.1 and Theorem 1.1.2. The arguments we use to prove these two theorems are similar. We start by giving in details the proof of Theorem 1.1.2.

**Definition 1.3.1.** *Let $(M^m, g)$ be a Riemannian manifold of dimension $m$. The capacity of a capacitor $(F, G)$ in $M$ is defined by*

$$\operatorname{cap}_g(F, G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^2 d\mu_g,$$

*where $\mathcal{T} = \mathcal{T}(F, G)$ is the set of all compactly supported Lipschitz functions on $M$ such that* $\operatorname{supp} \varphi \subset \mathrm{G}^\circ = \mathrm{G} \setminus \partial\mathrm{G}$ *and $\varphi \equiv 1$ in a neighborhood of $F$. If $\mathcal{T}(F, G)$ is empty, then* $\operatorname{cap}_g(F, G) = +\infty$. *Similarly, we can define the $m$-capacity as*

$$\operatorname{cap}^{(m)}_{[g]}(F, G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^m d\mu_g.$$

*Since m is the dimension of $M$, it is clear that the $m$-capacity depends only on the conformal class [g] of the metric $g$.*

**Proposition 1.3.1.** *Under the assumptions of Theorem 1.1.2, take the $m - m$ space $(\Omega, d_{g_0}, \mu)$, where $d_{g_0}$ is the Riemannian distance corresponding to the metric $g_0$ and $\mu$ is a non-atomic finite measure on $\Omega$. Then for every $n \in \mathbb{N}^*$, there exists a family of capacitors $\mathcal{A} = \{(F_i, G_i)\}_{i=1}^n$ with the following properties:*

*(i)* $\mu(F_i) \geq \frac{\mu(\Omega)}{8c_m^2 n}$ *;*

*(ii) the $G_i$'s are mutually disjoint ;*

*(iii) the family $\mathcal{A}$ is such that either*

> *(a) all the $F_i$'s are annuli, $G_i = 2F_i$ and $\mathrm{cap}^{(m)}_{[g_0]}(F_i, 2F_i) \le Q_m$, or*
>
> *(b) all the $F_i$'s are domains in $\Omega$ and $G_i = F_i^{r_0}$,*

*where $r_0 = \frac{1}{1600}$ and, $c_m$ and $Q_m$ are constants depending only on the dimension.*

*Proof.* Let us start with the observation that the metric space $(\Omega, d_{g_0})$ satisfies the $(2, N; 1)$-covering property. For each ball $B(x, r)$ with center in $\Omega$ and radius smaller than 1, take a maximal family $\{B(x_i, r/4)\}$ of disjoint balls with centers in $B(x, r)$. Let $\kappa$ be the cardinal of that family. The family of balls $\{B(x_i, r/2)\}$ covers $B(x, r)$. Hence

$$\kappa \min_i \mu_{g_0}(B(x_i, r/4)) \le \sum_i \mu_{g_0}(B(x_i, r/4)) \le \mu_{g_0}(B(x, r + r/4)).$$

Take $x_{i_0}$ such that $\mu_{g_0}(B(x_{i_0}, r/4)) = \min_i \mu_{g_0}(B(x_i, r/4))$. We have

$$\kappa \le \frac{\mu_{g_0}(B(x, r + r/4))}{\min_i \mu_{g_0}(B(x_i, r/4))} \le \frac{\mu_{g_0}(B(x, 2r))}{\mu_{g_0}(B(x_{i_0}, r/4))} \le \frac{\mu_{g_0}(B(x_{i_0}, 4r))}{\mu_{g_0}(B(x_{i_0}, r/4))}.$$

Since $\mathrm{Ricci}_{g_0}(\Omega) \ge -(m-1)$, thanks to the Bishop-Gromov volume comparison Theorem, we have $\forall\ 0 < s < r$,

$$\frac{\mu_{g_0}(B(x, r))}{\mu_{g_0}(B(x, s))} \le \frac{\int_0^r \sinh^{m-1} t \, dt}{\int_0^s \sinh^{m-1} t \, dt}.$$

Since for every positive $t$ one has $t \le \sinh t \le te^t$, we get

$$\frac{\mu_{g_0}(B(x, r))}{\mu_{g_0}(B(x, s))} \le \left(\frac{r}{s}\right)^m e^{(m-1)r}.$$

In particular, we have

$$\mu_{g_0}(B(x, r)) \le r^m e^{(m-1)r} \tag{1.15}$$

and, $\forall r < 1$,

$$\kappa \le \frac{\mu_{g_0}(B(x_{i_0}, 4r))}{\mu_{g_0}(B(x_{i_0}, r/4))} \le 2^{4m} e^{4(m-1)r} =: C(r) \le C(1). \tag{1.16}$$

One can take $N = C(1)$ and deduce that $(\Omega, d_{g_0})$ has the $(2, N; 1)$ covering property where $N$ depends only on the dimension.
Now the proof of Proposition 1.3.1 is a straightforward consequence of Theorem 1.2.1. Recall that in the statement of Theorem 1.2.1, the constant $c$ depends only on $N$. Therefore, in our case $c$ depends only on the dimension. It remains to verify that in the case of annuli, there exists a constant $Q_m$ depending only on the dimension such that for each $i$, we have

$$\mathrm{cap}^{(m)}_{[g_0]}(F_i, 2F_i) \le Q_m.$$

According to Theorem 1.2.1, the outer radii of the annuli we consider are smaller than one. It is enough to show that for each point $x \in \Omega$ and $0 \le r < R \le 1/2$, we have

$$\mathrm{cap}^{(m)}_{[g_0]}(A, 2A) \le Q_m, \tag{1.17}$$

where $A = A(x,r,R)$. Let $0 < r < R \leq 1/2$ and set

$$f(y) = \begin{cases} 1 & \text{if} \quad y \in A(x,r,R) \\ \frac{2d_{g_0}(y,B(x,r/2))}{r} & \text{if} \quad y \in A(x,r/2,r) = B(x,r) \setminus B(x,r/2) \\ 1 - \frac{d_{g_0}(y,B(x,R))}{R} & \text{if} \quad y \in A(x,R,2R) = B(x,2R) \setminus B(x,R) \\ 0 & \text{if} \quad y \in M \setminus A(x,r/2,2R) \end{cases}. \tag{1.18}$$

It is clear that $f \in \mathcal{T}(A, 2A)$ and

$$|\nabla_{g_0} f| \leq \frac{2}{r}, \quad \text{on } B(x,r) \setminus B(x,r/2),$$

$$|\nabla_{g_0} f| \leq \frac{1}{R}, \quad \text{on } B(x,2R) \setminus B(x,R).$$

Therefore

$$\begin{aligned} \operatorname{cap}^{(m)}_{[g_0]}(A,2A) &\leq \int_M |\nabla_{g_0} f|^m d\mu_{g_0} \\ &\leq \big(\frac{2}{r}\big)^m \mu_{g_0}(A(x,r/2,r)) + \big(\frac{1}{R}\big)^m \mu_{g_0}(A(x,R,2R)) \\ &\leq \big(\frac{2}{r}\big)^m \mu_{g_0}(B(x,r)) + \big(\frac{1}{R}\big)^m \mu_{g_0}(B(x,2R)). \end{aligned}$$

Now since $r, 2R \in (0,1]$, Using Inequality (1.15), one can control the last inequality by a constant $Q_m$ depending only on the dimension. If $r = 0$, we set

$$f(y) = \begin{cases} 1 & \text{if} \quad y \in B(x,R) \\ 1 - \frac{d_{g_0}(y,B(x,R))}{R} & \text{if} \quad y \in B(x,2R) \setminus B(x,R) \\ 0 & \text{if} \quad y \in M \setminus B(x,2R) \end{cases}. \tag{1.19}$$

And the same argument as above shows that the capacity can be controlled by a constant $Q_m$ depending only on the dimension which completes the proof of Inequality (1.17). □

**Remark 1.3.1.** *Since $C(r)$ defined in (1.16) is a strictly increasing function of $r$, it follows that $(\Omega, d_{g_0})$ does not necessarily satisfy the $(2,N)$-covering property for some $N$ depending only on the dimension.*

Now we show how Theorem 1.1.2 follows from Proposition 1.3.1.

*Proof of Theorem 1.1.2.* Take the $m-m$ space $(\Omega, d_{g_0}, \mu_\Omega)$, where $\mu_\Omega = \mu_g|_\Omega$. According to Proposition 1.3.1, there exists a family $\{(F_i, G_i)\}$ of $3k$ capacitors which satisfies the properties $(i), (ii)$ and either $(iii)(a)$ or $(iii)(b)$ of the proposition. We consider each case separately.

**Case 1.** If $\{(F_i, G_i)\}_{i=1}^{3k}$ is a family with the properties $(i), (ii)$ and $(iii)(a)$ of Proposition 1.3.1, then

$$\lambda_k(\Omega, g) \leq A'_m \left(\frac{k}{\mu_g(\Omega)}\right)^{\frac{2}{m}}, \tag{1.20}$$

where $A'_m = 24c_m^2(2Q_m)^{\frac{2}{m}}$.

Indeed, we begin by choosing a family of $3k$ test functions $\{f_i : f_i \in \mathcal{T}(F_i, G_i)\}_{i=1}^{3k}$ such that

$$\int_M |\nabla_{g_0} f_i|^m d\mu_{g_0} \le \mathrm{cap}^{(m)}_{[g_0]}(F_i, G_i) + \epsilon.$$

Therefore,

$$\begin{aligned} R(f_i) &= \frac{\int_\Omega |\nabla_g f_i|^2 d\mu_g}{\int_\Omega |f_i|^2 d\mu_g} \le \frac{\left(\int_\Omega |\nabla_{g_0} f_i|^m d\mu_{g_0}\right)^{\frac{2}{m}} \left(\int_\Omega 1_{\mathrm{supp} f_i} d\mu_g\right)^{1-\frac{2}{m}}}{\int_\Omega |f_i|^2 d\mu_g} \\ &\le \frac{\left(\mathrm{cap}^{(m)}_{[g_0]}(F_i, G_i) + \epsilon\right)^{\frac{2}{m}} \left(\mu_\Omega(G_i)\right)^{1-\frac{2}{m}}}{\mu_\Omega(F_i)}. \end{aligned} \tag{1.21}$$

The first inequality follows from Hölder inequality and, because of the conformal invariance of $\int |\nabla_g f_i|^m d\mu_g$, we have replaced $g$ by $g_0$. Since the $G_i$'s are disjoint domains and $\sum_{i=1}^{3k} \mu_\Omega(G_i) \le \mu_g(\Omega)$, at least $k$ of them have measure smaller than $\frac{\mu_g(\Omega)}{k}$. Up to re-ordering, we assume that for the first $k$ of the $G_i$'s we have

$$\mu_\Omega(G_i) \le \frac{\mu_g(\Omega)}{k}. \tag{1.22}$$

Now, we can take $\epsilon = Q_m$. Using Proposition 1.3.1 (i) and (iii)(a) and Inequality (1.22), we get from Inequality (1.21)

$$R(f_i) \le A'_m \frac{\left(\frac{\mu_g(\Omega)}{k}\right)^{1-\frac{2}{m}}}{\frac{\mu_g(\Omega)}{k}} = A'_m \left(\frac{k}{\mu_g(\Omega)}\right)^{\frac{2}{m}},$$

with $A'_m = 24 c_m^2 (2Q_m)^{\frac{2}{m}}$, which completes the proof of Case 1.

**Case 2.** If $\{(F_i, G_i)\}_{i=1}^{3k}$ is a family with the properties $(i), (ii)$ and $(iii)(b)$ of Proposition 1.3.1, then

$$\lambda_k(\Omega, g) \le B'_m \left(\frac{\mu_{g_0}(\Omega)}{\mu_g(\Omega)}\right)^{\frac{2}{m}}, \tag{1.23}$$

where $B'_m = \frac{24 c_m^2}{r_0^2}$.

Indeed, we define the test functions $f_i$ as follows

$$f_i(x) = \begin{cases} 1 & \text{if} \quad x \in F_i \\ 1 - \frac{d_{g_0}(x, F_i)}{r_0} & \text{if} \quad x \in (G_i \setminus F_i) \\ 0 & \text{if} \quad x \in G_i^c \end{cases}.$$

We have $|\nabla_{g_0} f_i| \le \frac{1}{r_0}$. Therefore,

$$\begin{aligned} R(f_i) &= \frac{\int_\Omega |\nabla_g f_i|^2 d\mu_g}{\int_M |f_i|^2 d\mu_g} \le \frac{\left(\int_\Omega |\nabla_{g_0} f_i|^m d\mu_{g_0}\right)^{\frac{2}{m}} \left(\int_\Omega 1_{\mathrm{supp} f_i} d\mu_g\right)^{1-\frac{2}{m}}}{\int_\Omega |f_i|^2 d\mu_g} \\ &\le \frac{\frac{1}{r_0^2} \left(\mu_{g_0}(G_i \cap \Omega)\right)^{\frac{2}{m}} \left(\mu_\Omega(G_i)\right)^{1-\frac{2}{m}}}{\mu_\Omega(F_i)}. \end{aligned} \tag{1.24}$$

Since the $G_i$'s are disjoint, we have

$$\sum_{i=1}^{3k} \mu_{g_0}(G_i \cap \Omega) \leq \mu_{g_0}(\Omega) \quad \text{and} \quad \sum_{i=1}^{3k} \mu_\Omega(G_i) \leq \mu_g(\Omega).$$

Hence, there exist at least $2k$ sets among $G_1, \dots, G_{3k}$ such that $\mu_{g_0}(G_i) \leq \frac{\mu_{g_0}(\Omega)}{k}$. Similarly, there exist at least $2k$ sets (not necessarily the same ones) such that $\mu_g(G_i) \leq \frac{\mu_g(\Omega)}{k}$. Therefore, up to re-ordering, we assume that the first $k$ of the $G_i$'s satisfy both of the two following inequalities

$$\mu_\Omega(G_i) \leq \frac{\mu_g(\Omega)}{k} \quad \text{and} \quad \mu_{g_0}(G_i \cap \Omega) \leq \frac{\mu_{g_0}(\Omega)}{k}. \tag{1.25}$$

Using Proposition 1.3.1 ($i$) and Inequalities (1.25), we get from Inequality (1.24)

$$\begin{aligned} R(f_i) &\leq B'_m \frac{\left(\frac{\mu_{g_0}(\Omega)}{k}\right)^{\frac{2}{m}} \left(\frac{\mu_g(\Omega)}{k}\right)^{1-\frac{2}{m}}}{\frac{\mu_g(\Omega)}{k}} \\ &\leq B'_m \left(\frac{\mu_{g_0}(\Omega)}{\mu_g(\Omega)}\right)^{\frac{2}{m}} \end{aligned}$$

with $B'_m = \frac{24c_m^2}{r_0^2}$, which completes the proof of Case 2.

In both cases, $\lambda_k(\Omega, g)$ is bounded above by the sum of the right-hand sides of (1.20) and (1.23), which completes the proof. $\square$

**Remark 1.3.2.** To avoid a possible confusion, it is judicious to examine the proof of Theorem 1.1.2. In the proof, we begin with the GNY-construction but the method may break down for some $j < 2n$ in the sense that we may not be able to find $j$ (or more) disjoint small annuli. In such a case, Inequality (1.14) holds. The validity of Inequality (1.14) implies that the CM-construction is applicable with $r = r_0$ which gives an estimate for $\lambda_k$ of the form given in Inequality (1.23). This may appear to be unreasonable since the right hand side is independent of $k$. However, as pointed out in Proposition 1.2.1, the GNY-construction for a given compact Riemannian manifold is applicable for all $n$ sufficiently large, but we have no control over the constants and how large $n$ should be. The method described above enables one to establish the validity of the estimate for those finite number of $k$'s for which the GNY-construction is not applicable.

*Proof of Theorem 1.1.1.* Consider the $m-m$ space $(M, d_{g_0}, \mu_g)$, where $d_{g_0}$ is the distance associated with the metric $g_0$ and $\mu_g$ is the measure associated with the metric $g$. We easily see that we can follow the same arguments as in the proof of Theorem 1.1.2 to derive the following inequality

$$\lambda_k(M, g)\mu_g(M)^{\frac{2}{m}} \leq A_m \mu_{g_0}(M)^{\frac{2}{m}} + B_m k^{\frac{2}{m}}. \tag{1.26}$$

The left hand side does not depend on $g_0$. Hence, we can take the infimum with respect to $g_0 \in [g]$ such that $\mathrm{Ricci}_{g_0} \geq -(m-1)$, which leads to the desired conclusion. $\square$

## 1.4 Steklov Eigenvalues

It is worth pointing out that Theorem 1.2.1 is formalized in a general setting and is applicable to other eigenvalue problems. In this section we present an application of this theorem to the Steklov eigenvalue problem.

**Steklov problem.** Let $\Omega$ be a bounded subdomain of a complete $m$-dimensional Riemannian manifold $(M, g)$ and assume that $\Omega$ has nonempty smooth boundary $\partial\Omega$. Given a function $u \in H^{\frac{1}{2}}(\partial\Omega)$, we denote by $\bar{u}$ the unique harmonic extension of $u$ to $\Omega$, that is

$$\begin{cases} \Delta_g \bar{u} = 0 & \text{in } \Omega \\ \bar{u} = u & \text{on } \partial\Omega \end{cases}.$$

Let $\nu$ be the outward unit normal vector along $\partial\Omega$. The Steklov operator is the map

$$\begin{array}{rcl} L : H^{\frac{1}{2}}(\partial\Omega) & \to & H^{-\frac{1}{2}}(\partial\Omega) \\ u & \mapsto & \dfrac{\partial \bar{u}}{\partial \nu}. \end{array}$$

The operator $L$ is an elliptic pseudo differential operator (see [38, pages 37-38]) which admits a discrete spectrum tending to infinity denoted by

$$0 = \sigma_1 \le \sigma_2 \le \sigma_3 \ldots \nearrow \infty$$

The eigenvalue $\sigma_k$ of $L$ can be characterized variationally as follows (see [14]):

$$\sigma_k(\Omega) = \inf_{V_k} \sup\{ \frac{\int_\Omega |\nabla_g \bar{u}|^2 d\mu_g}{\int_{\partial\Omega} |\bar{u}|^2 d\bar{\mu}_g} : 0 \ne \bar{u} \in V_k \}, \tag{1.27}$$

where $V_k$ is a $k$-dimensional linear subspace of $H^1(\Omega)$ and $\bar{\mu}_g$ is the Riemannian measure associated to $g$ on the boundary.

The relationships between the geometry of the domain and the spectrum of the corresponding Steklov operator have been investigated by several authors (see for example [14], [19] and [21]). Recently, Fraser and Schoen [19, Theorem 2.3] proved the following inequality for the Steklov eigenvalues of a compact Riemannian surface $(\Sigma_\gamma, g)$ of genus $\gamma$ and $\kappa$ boundary components:

$$\sigma_2(\Sigma_\gamma)\ell_g(\partial\Sigma_\gamma) \le 2(\gamma + \kappa)\pi,$$

where $\ell_g(\partial\Sigma)$ is the length of the boundary. This result was generalized to higher eigenvalues by Colbois, El Soufi and Girouard [14, Theorem 1.5]. Indeed, the authors proved the following inequality for every $k \in \mathbb{N}^*$

$$\sigma_k(\Sigma_\gamma)\ell_g(\partial\Sigma_\gamma) \le C(\gamma + 1)k, \tag{1.28}$$

where $C$ is a universal constant.
For a domain in a higher dimensional manifold, the authors [14, Theorem 1.3] also obtained

an upper bound for $\sigma_k$ depending on the isoperimetric ratio of the domain. More precisely, if $(M,g)$ is conformally equivalent to a complete manifold with non-negative Ricci curvature, then for every bounded domain $\Omega$ of $M$ and every $k \in \mathbb{N}^*$,

$$\sigma_k(\Omega)\bar{\mu}_g(\partial\Omega)^{\frac{1}{m-1}} \leq C_m \frac{k^{\frac{2}{m}}}{I_g(\Omega)^{1-\frac{1}{m-1}}}, \tag{1.29}$$

where $I_g(\Omega)$ is the isoperimetric ratio ($I_g(\Omega) = \frac{\bar{\mu}_g(\partial\Omega)}{\mu_g(\Omega)^{\frac{m-1}{m}}}$) and $C_m$ is a constant depending only on $m$.

The theorem below is motivated by the work of [14], and we obtain an improvement of Inequalities (1.28) and (1.29) using Proposition 1.3.1.

**Theorem 1.4.1.** *Let $(M, g_0)$ be a complete Riemannian manifold of dimension $m \geq 2$ with $\mathrm{Ricci}_{g_0}(M) \geq -(m-1)$. Let $\Omega \subset M$ be a relatively compact domain with $C^1$ boundary and $g$ be any metric conformal to $g_0$. Then we have*

$$\sigma_k(\Omega)\bar{\mu}_g(\partial\Omega)^{\frac{1}{m-1}} \leq \frac{A_m\mu_{g_0}(\Omega)^{\frac{2}{m}} + B_m k^{\frac{2}{m}}}{I_g(\Omega)^{1-\frac{1}{m-1}}}, \tag{1.30}$$

*where $A_m$ and $B_m$ are constants depending only on $m$.*

As an immediate consequence we get the following inequality in the case of Riemann surfaces:

**Corollary 1.4.1.** *Let $(\Sigma_\gamma, g)$ be a compact oriented Riemannian surface with genus $\gamma$, and $\Omega$ be a subdomain of $\Sigma_\gamma$. Then*

$$\sigma_k(\Omega)\ell_g(\partial\Omega) \leq A\gamma + Bk,$$

*where $A$ and $B$ are constants.*

*Proof of Theorem 1.4.1.* We consider the $m-m$ space $(\Omega, d_{g_0}, \bar{\mu})$, where $\bar{\mu}(A) := \bar{\mu}_g(A \cap \partial\Omega)$. We apply again Proposition 1.3.1. Therefore, there exist a family of $3k$ capacitors $\{(F_i, G_i)\}$ satisfying properties $(i), (ii)$ and either $(iii)(a)$, or $(iii)(b)$ of Proposition 1.3.1. We proceed analogously to the proof of Theorem 1.1.2. Using the variational characterization of $\sigma_k$, we construct a family of test functions as in Case 1 and Case 2 of the proof of Theorem 1.1.2. In both cases, we have

$$\sigma_k(\Omega) \leq \frac{\int_\Omega |\nabla_g f_i|^2 d\mu_g}{\int_{\partial\Omega} |f_i|^2 d\bar{\mu}_g} \leq \frac{\left(\int_\Omega |\nabla_{g_0} f_i|^m d\mu_{g_0}\right)^{\frac{2}{m}} \mu_g(G_i)^{1-\frac{2}{m}}}{\bar{\mu}(F_i)}.$$

If the family $\{(F_i, G_i)\}$ satisfies the properties $(i), (ii)$ and $(iii)(a)$ of Proposition 1.3.1, then

$$\sigma_k(\Omega) \leq A_m \frac{\left(\frac{\mu_g(\Omega)}{k}\right)^{1-\frac{2}{m}}}{\frac{\bar{\mu}_g(\partial\Omega)}{k}} \leq A_m \frac{k^{\frac{2}{m}}}{\bar{\mu}_g(\partial\Omega)^{\frac{1}{m-1}} I_g(\Omega)^{1-\frac{1}{m-1}}}. \tag{1.31}$$

If on the other hand, the family $\{(F_i, G_i)\}$ satisfies the properties $(i), (ii)$ and $(iii)(b)$ of Proposition 1.3.1, then

$$\sigma_k(\Omega) \leq B_m \frac{\left(\frac{\mu_{g_0}(\Omega)}{k}\right)^{\frac{2}{m}} \left(\frac{\mu_g(\Omega)}{k}\right)^{1-\frac{2}{m}}}{\frac{\bar{\mu}_g(\partial\Omega)}{k}} \leq B_m \frac{\mu_{g_0}(\Omega)^{\frac{2}{m}}}{\bar{\mu}_g(\partial\Omega)^{\frac{1}{m-1}} I_g(\Omega)^{1-\frac{1}{m-1}}}, \tag{1.32}$$

where the constant coefficients $A_m$ and $B_m$ are the same as $A'_m$ and $B'_m$ in Theorem 1.1.2. The proof of Inequalities (1.31) and (1.32) are along the same lines as Theorem 1.1.2. In both cases, $\sigma_k(\Omega)$ is bounded above by the sum on the right-hand sides of (1.31) and (1.32), and it completes the proof. $\square$

# Chapitre 2

# Valeurs propres du Laplacien et géométrie extrinsèque

# Eigenvalues of the Laplacian and extrinsic geometry

## 2.1 Introduction

The investigation of the relationships between the extrinsic geometry of submanifolds and the spectrum of the Laplace-Beltrami operator is an important topic of spectral geometry. The purpose of this chapter, inspired by the recent work of Colbois, Dryden and El Soufi, is to give upper bounds for the eigenvalues of the Laplacian in terms of some extrinsic data. Colbois, Dryden and El Soufi [12] study the relation between an extrinsic invariant of submanifolds of $\mathbb{R}^N$, called the *intersection index*, and the eigenvalues of the Laplace-Beltrami operator. For a compact $m$-dimensional immersed submanifold $M$ of $\mathbb{R}^N = \mathbb{R}^{m+p}$, $p > 0$, the *intersection index* is given by

$$i(M) = \sup_{\Pi} \sharp(M \cap \Pi),$$

where $\Pi$ runs over the set of all $p$-planes that are transverse to $M$ ; if $M$ is not embedded, we count multiple points of $M$ according to their multiplicity. We remark that the intersection index was also investigated by Thom [7] where it was called the *degree* of $M$.
In [12], Colbois, Dryden and El Soufi show that there is a positive constant $c(m)$, depending only on $m$ such that for every compact $m$-dimensional immersed submanifold of $\mathbb{R}^{m+p}$, we have the following inequality

$$\lambda_k(M)\mathrm{Vol}(M)^{2/m} \le c(m)i(M)^{2/m}k^{2/m}. \tag{2.1}$$

Moreover, the intersection index in the above inequality is not replaceable with a constant depending only on the dimension $m$. Even for hypersurfaces, the first positive eigenvalue cannot be controlled only in terms of the volume and the dimension (see [12, Theorem 1.4]).
An immediately consequence of Inequality (2.1) is the fact that for convex hypersurfaces, the normalized eigenvalues are bounded above only in terms of the dimension. Another remarkable consequence of Inequality (2.1) concerns algebraic submanifolds [12, Corollary 4.1]: Let $M$ be a compact real algebraic manifold, i.e. $M$ is a zero locus of $p$ real polynomials in $m + p$ variables of degrees $N_1, \dots, N_p$. Then

$$\lambda_k(M)\mathrm{Vol}(M)^{2/m} \le c(m)N_1^{2/m} \cdots N_p^{2/m}k^{2/m}. \tag{2.2}$$

Notice that Inequality (2.1) is not stable under "*small*" perturbations, since the intersection index might dramatically change.

In this chapter, we extend the work of Colbois, Dryden and El Soufi in two directions. The first one consists in replacing the intersection index $i(M)$ by locally defined invariants of the same nature which are stable under "*small*" perturbations. The second direction concerns complex submanifolds of the complex projective space $\mathbb{C}P^N$.

**First part.** Let $\varepsilon$ be a positive number, $\varepsilon < 1$. By "$\varepsilon$-small" perturbation we mean any perturbation in a region $D \subset M$ whose measure is at most equal to $\varepsilon$Vol(M). To avoid any technical complexity, we assume that $M \setminus D$ is a smooth manifold with smooth boundary. Let us define these new notions of intersection indexes. Let $G$ be the Grassmannian of all $m$-vector spaces in $\mathbb{R}^{m+p}$ endowed with an invariant Haar measure with total measure 1. Let $0 < \varepsilon < 1$ and $D$ be any open subdomain of $M$ such that $M \setminus D$ is a smooth manifold with smooth boundary and $\mathrm{Vol}(D) \leq \varepsilon \mathrm{Vol}(M)$. We denote $M \setminus D$ by $M_\varepsilon^D$ and $\sup_{P \perp H} \sharp(M_\varepsilon^D \cap P)$ by $i_H(M_\varepsilon^D)$, where $P$ is an affine $p$-plane orthogonal to $H$. We now define the *$\varepsilon$-mean intersection index* as follows:

$$\bar{\imath}^\varepsilon(M) := \inf_D \int_G i_H(M_\varepsilon^D) dH,$$

where $D$ runs over regions whose measure is smaller than $\varepsilon \mathrm{Vol}(M)$ and $M \setminus D$ is a smooth manifold with smooth boundary.
Similarly, for $r > 0$, we define the *$(r, \varepsilon)$-local intersection index* as:

$$\bar{\imath}_r^\varepsilon(M) := \inf_D \sup_{x \in M_\varepsilon^D} \int_G i_H(M_\varepsilon^D \cap B(x,r)) dH,$$

where $B(x,r) \subset \mathbb{R}^{m+p}$ is an Euclidean ball centered at $x$ and of radius $r$ and $D$ runs over regions whose measure is smaller than $\varepsilon \mathrm{Vol}(M)$ and $M \setminus D$ is a smooth manifold with smooth boundary.
We can now state our theorem.

**Theorem 2.1.1.** *There exist positive constants $c_m, \alpha_m$ and $\beta_m$ depending only on $m$ such that for every compact $m$-dimensional immersed submanifold $M$ of $\mathbb{R}^{m+p}$ and every $k \in N^*$ and $\varepsilon > 0$, we have*

$$\lambda_k(M)\, Vol(M)^{2/m} \leq c_m \frac{\bar{\imath}^\varepsilon(M)^{2/m}}{(1-\varepsilon)^{1+2/m}} k^{2/m}, \tag{2.3}$$

*and*

$$\lambda_k(M) \leq \alpha_m \frac{1}{(1-\varepsilon) r^2} + \beta_m \frac{\bar{\imath}_r^\varepsilon(M)^{2/m}}{(1-\varepsilon)^{1+2/m}} \left( \frac{k}{Vol(M)} \right)^{2/m}. \tag{2.4}$$

The main feature of the inequalities of Theorem 2.1.1 is the fact that the upper bounds are not considerably affected by the presence of a large intersection index in a "small" part of $M$ (i.e. a subdomain with small volume). In particular, for a compact hypersurface of $\mathbb{R}^{m+1}$ which is convex outside[1] a region $D$ of measure at most $\varepsilon$Vol(M), one has $\bar{\imath}^\varepsilon(M) \leq i(M_D^\varepsilon)$ and then

$$\lambda_k(M) \mathrm{Vol}(M)^{2/m} \leq c_m \frac{2^{2/m}}{(1-\varepsilon)^{1+2/m}} k^{2/m}.$$

[1]We say that $M$ is convex outside of $D$ if after a perturbation of $M$ which is the identity outside of $D$ we get a convex compact hypersurface.

**Second part.** We study another natural context where algebraic submanifolds can be considered which is the complex projective space $\mathbb{C}P^N$. According to Chow's Theorem, every complex submanifold $M$ of $\mathbb{C}P^N$ is a smooth algebraic variety, i.e. it is a zero locus of a family of complex polynomials. We obtain the following upper bound for complex submanifolds of $\mathbb{C}P^N$ endowed with Fubini-Study metric $g_{_{FS}}$.

**Theorem 2.1.2.** *Let $M^m$ be an $m$-dimensional complex manifold admitting a holomorphic immersion $\phi : M \to \mathbb{C}P^N$. Then for every $k \in \mathbb{N}^*$ we have*

$$\lambda_{k+1}(M, \phi^* g_{_{FS}}) \leq 2(m+1)(m+2)k^{\frac{1}{m}} - 2m(m+1). \tag{2.5}$$

In particular, one has Inequality (2.5) for every complex submanifold of $\mathbb{C}P^N$. Note that the power of $k$ is compatible with the Weyl law.
Under the assumption of the above theorem, for $k = 1$, one has

$$\lambda_2(M, \phi^* g_{_{FS}}) \leq 4(m+1), \tag{2.6}$$

which is a sharp inequality since the equality holds for $\mathbb{C}P^m$. This sharp upper bound was also obtained by Bourguignon, Li and Yau [4, page 200] (see also the paper by Arezzo, Ghigi and Loi [1]). Theorem 2.1.2 gives us another proof of this sharp inequality ; moreover, we do not need any restriction on holomorphic immersions (see page 43).
For a complex submanifold $M$ of $\mathbb{C}P^{m+p}$ of the complex dimension $m$, we have

$$\mathrm{Vol(M)} = \deg(M)\mathrm{Vol}(\mathbb{C}P^m),$$

where $\deg(M)$ is the intersection number of $M$ with a projective $p$-plane in a generic position. Moreover, one can describe $M$ as a zero locus of a family of irreducible homogenous polynomials which the $\deg(M)$ is the multiplication of degrees of the irreducible polynomials that describe $M$ (see for example [23, pages 171-172]). Therefore, we can rewrite Inequality (2.5) as

$$\lambda_{k+1}(M, g_{_{FS}})\mathrm{Vol}(M)^{\frac{1}{m}} = \lambda_{k+1}(M, g_{_{FS}}) \left(\deg(M)\mathrm{Vol}(\mathbb{C}P^m)\right)^{\frac{1}{m}} \leq C(m) \deg(M)^{\frac{1}{m}} k^{\frac{1}{m}}. \tag{2.7}$$

One can now compare Inequality (2.7) with Inequality (2.2).
This chapter is organized as follows: In section 2.2, we present a more general and abstract setting and we illustrate applications of it in Section 2.3 where we prove Theorem 2.1.1. In Section 2.4, we consider algebraic submanifolds of $\mathbb{C}P^N$ and we prove Theorem 2.1.2. The method which is used in Section 2.4 to show Theorem 2.1.2 is independent from what we introduce in Sections 2.2 and 2.3.

## 2.2 A general preliminary result

This section is devoted to introduce a more general and abstract setting. It is another illustration of the metric construction introduced by Colbois and Maerten [16]. They introduced a metric approach to construct an elaborated family of disjoint domains in a metric-measure space ($m - m$ space) $X$, with certain properties. The following theorem

relies on this construction (see [15, Lemma 2.1] or [16, Corollary 2.3]).
Throughout this section the triple $(X, d, \mu)$ will designate a complete locally compact $m-m$ space with a distance $d$ and a finite and positive non-atomic Borel measure $\mu$. We also assume that balls in $(X, d)$ are pre-compact. In the sequel, we use the notations and definitions introduced in the previous chapter.
We define the *dilatation* of a function $f : (X, d) \to \mathbb{R}$ as

$$\mathrm{dil}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)},$$

and the *local dilatation* at $x \in X$ as

$$\mathrm{dil}_x(f) = \lim_{\varepsilon \to 0} \mathrm{dil}(f|_{B(x,\varepsilon)}).$$

When different distance functions are considered, $\mathrm{dil}_d(f)$ and $\mathrm{dil}_{d,x}(f)$ stand for the dilatation and local dilatation at $x$ associated with the distance $d$.
A map is called Lipschitz if $\mathrm{dil}(f) < \infty$. Let $(M, g)$ be a Riemannian manifold and $d_g$ be the distance associated to the Riemannian metric $g$. A Lipschitz function on a Riemannian manifold $M$ is differentiable almost everywhere and $|\nabla_g f(x)|$ coincides with $\mathrm{dil}_x(f)$ almost everywhere. Hence, $|\nabla_g f(x)| \leq \mathrm{dil}(f)$ almost everywhere.
Each pair $(F, G)$ of Borel sets in $X$ such that $F \subset G$ is called a capacitor. Given a capacitor $(F, G)$, let $\mathcal{T}(F, G)$ be the set of all compactly supported real valued functions on $X$ so that for every $\varphi \in \mathcal{T}(F, G)$ we have supp $\varphi \subset \mathrm{G}^\circ = \mathrm{G} \setminus \partial\mathrm{G}$ and $\varphi \equiv 1$ in a neighborhood of $F$.
The following theorem gives a construction of a family of disjointly supported functions with a nice control on their dilatations provided some conditions on the metric-measure structure are imposed. We will see its application in Corollary 2.2.1.

**Theorem 2.2.1.** *Let positive constants $p, \rho, L$ and $N$ be given and $(X, d, \mu)$ be an $m - m$ space satisfying the $(4, N; \rho)$-covering property and*

$$\mu(B(x, r)) \leq Lr^p, \quad \textit{for every } x \in X \textit{ and } 0 < r \leq \rho.$$

*Then for every $n \in \mathbb{N}^*$ and every $r \leq \min\{\rho, \left(\frac{\mu(X)}{4N^2Ln}\right)^{1/p}\}$ there is a family of $n$ mutually disjoint bounded capacitors $\{(A_i, A_i^r)\}_{i=1}^n$, of $X$ and a family $\{f_i\}$ of $n$ Lipschitz functions with $f_i \in \mathcal{T}(A_i, A_i^r)$ such that $\mu(A_i) \geq \frac{\mu(X)}{2Nn}$ and*

$$\mathrm{dil}_d(f_i) \leq \frac{1}{\rho} + (4N^2L)^{1/p} \left(\frac{n}{\mu(X)}\right)^{1/p}. \tag{2.8}$$

If the condition $\mu(B(x, r)) \leq Lr^p$ is satisfied for every $r > 0$ then we take $\rho = \infty$. Hence, the first term at the right-hand side of the above inequality vanishes.

*Proof of Theorem 2.2.1.* According to Colbois and Maerten's result (see Lemma 1.2.3), if the $m - m$ space $(X, d, \mu)$ has $(4, N; \rho)$-covering property, then for every $r \leq \rho$ such that

$$\mu(B(x, r)) \leq \frac{\mu(X)}{4N^2n}, \quad \forall x \in X, \tag{2.9}$$

we have a family $\{(A_i, A_i^r)\}$ of mutually disjoint capacitors of $X$ with the desired property mentioned in the theorem. We claim that when $r \leq \min\{\rho, \left(\frac{\mu(X)}{4N^2Ln}\right)^{1/p}\}$, the Inequality (2.9) is automatically satisfied. Indeed, according to the assumptions we have

$$\mu(B(x,r)) \leq Lr^p \leq \min\{L\rho^p, \frac{\mu(X)}{4N^2n}\} \leq \frac{\mu(X)}{4N^2n}.$$

We now consider Lipschitz functions $f_i$'s supported on $A_i^r$ with $f_i(x) = 1 - \frac{d(x,A_i)}{r}$ on $A_i^r \setminus A_i$, $f_i(x) = 1$ on $A_i$ and zero outside of $A_i^r$. One can easily check that $\mathrm{dil}_d(f_i) \leq \frac{1}{r}$. Hence, we obtain:

$$\mathrm{dil}_d(f_i) \leq \frac{1}{\rho} + (4N^2L)^{1/p}\left(\frac{k}{\mu(X)}\right)^{1/p}.$$

This completes the proof. □

Let $(M, g, \mu)$ be a Riemannian manifold endowed with a finite non-atomic Borel measure $\mu$. We define the following quantity that coincides with the eigenvalues of the Laplace-Beltrami operator when $\mu$ coincides with the Riemannian measure $\mu_g$.

$$\lambda_k(M, g, \mu) := \inf_L \sup\{R(f) : f \in L\},$$

where $L$ is a $k$-dimensional vector space of Lipschitz functions and

$$R(f) = \frac{\int_M |\nabla_g f|^2 d\mu}{\int_M f^2 d\mu}$$

As an application of Theorem 2.2.1 in the Riemannian case, we have

**Corollary 2.2.1.** *Let $(M, g, \mu)$ be a Riemannian manifold with a finite non-atomic Borel measure $\mu$ and the distance $d_g$ associated to the Riemannian metric $g$. If there exists a measure $\nu$ and a distance $d$ so that*

$$d(x,y) \leq d_g(x,y), \quad \forall x, y \in M; \tag{2.10}$$

$$\nu(A) \leq \mu(A) \quad \textit{for all measurable subset } A \textit{ of } M, \tag{2.11}$$

*and moreover, there exist positive constants $p, \rho, N$ and $L$ so that $(M, d, \nu)$ satisfies the assumptions of Theorem 2.2.1, then, for every $n \in \mathbb{N}^*$ we have*

$$\lambda_k(M, g, \mu) \leq \frac{16N}{\rho^2} + 16N(4N^2L)^{2/m}\left(\frac{\mu(M)}{\nu(M)}\right)^{1+2/p}\left(\frac{k}{\mu(M)}\right)^{2/p}. \tag{2.12}$$

*Proof.* Take $(M, d, \nu)$ as an $m-m$ space. According to Theorem 2.2.1, for every $2k \in \mathbb{N}^*$, we have a family of $2k$ mutually disjoint capacitors $\{(F_i, G_i)\}_{i=1}^{2k}$ and $2k$ Lipschitz functions $f_i$'s such that for every $1 \leq i \leq 2k$, $\nu(F_i) \geq \frac{\nu(M)}{4Nk}$ and $\mathrm{dil}_d(f_i)$ satisfies Inequality (2.8). Here, we have $|\nabla_g f| \leq \mathrm{dil}_{d_g}(f)$ almost everywhere. Since $d \leq d_g$ and $\nu \leq \mu$ we get

$$|\nabla_g f| \leq \mathrm{dil}_{d_g}(f) \leq \mathrm{dil}_d(f) \leq \frac{1}{\rho} + (4N^2L)^{1/p}\left(\frac{2k}{\nu(M)}\right)^{1/p},$$

and

$$\mu(F_i) \geq \nu(F_i) \geq \frac{\nu(M)}{4Nk}.$$

Since the support of $f_i$'s are disjoint and $\sum_{i=1}^{2k}\mu(A_i^r) \leq \mu(M)$, at least $k$ of them have measure smaller than $\frac{\mu(M)}{k}$. Up to re-ordering, we assume that for the first $k$ of the $A_i^r$'s we have

$$\mu(A_i^r) \leq \frac{\mu(M)}{k}.$$

Therefore,

$$\begin{aligned}\lambda(M,g,\mu) \leq \max_i R(f_i) &\leq \max_i \left(\frac{1}{\rho} + (4N^2L)^{1/p}\left(\frac{2k}{\nu(M)}\right)^{1/p}\right)^2 \frac{\mu(A_i^r)}{\mu(A_i)}\\ &\leq 16N\left(\frac{1}{\rho^2} + (4N^2L)^{2/p}\left(\frac{2k}{\nu(M)}\right)^{2/p}\right)\frac{\mu(M)}{\nu(M)},\end{aligned}$$

and we obtain Inequality (2.12). □

## 2.3 Eigenvalues of Immersed Submanifolds of $\mathbb{R}^N$

In this section, we prove Theorem 2.1.1. Let $S$ be an $m$-dimensional immersed submanifold of $\mathbb{R}^{m+p}$ (with or without boundary). We recall that $G$ is the Grassmannian of all $m$-vector spaces in $\mathbb{R}^{m+p}$ endowed with an invariant Haar measure with total measure 1. We define the *mean intersection index* of $S$ as follows:

$$\bar{\imath}(S) := \int_G i_H(S)dH,$$

where $i_H(S) := \sup_{P\perp H} \sharp(S\cap P)$ and $P$ is an affine $p$-plane orthogonal to $H$.
Similarly, for every $r>0$, we define the *$r$-local intersection index* of $S$ by:

$$\bar{\imath}_r(S) := \sup_{x\in S}\int_G i_H(S\cap B(x,r))dH,$$

where $B(x,r)\subset\mathbb{R}^{m+p}$ is an Euclidean ball of radius $r$ centered at $x$.
Let $H\in G$ and $\pi_H : S\to H$ be the orthogonal projection of $S$ on $H$. The following lemma extends what is done in [12, Lemma 2.1].

**Lemma 2.3.1.** *Let $S$ be an $m$-dimensional immersed submanifold of $\mathbb{R}^{m+p}$, (not necessarily without boundary). Then we have*

$$Vol(S) \leq C_m\bar{\imath}(S)\mathrm{Vol}(\pi_{\mathrm{H}}(\mathrm{S})), \tag{2.13}$$

*where $C_m$ is a constant depending only on the dimension $m$ of $S$.*

*Proof.* Since for almost all $H\in G$, a point in $\pi_H(S)$ has finite number of preimages, one can take a generic $H$ and get

$$\int_S \pi_H^* v_H = \int_S |\theta_H(x)|v_S \leq \int_{\pi_H(S)} i_H(S)v_H = i_H(S)\mathrm{Vol}(\pi_H(S)),$$

where $v_S$ and $v_H$ are volume elements of $S$ and $H$ respectively and

$$|\theta_H(x)|v_S = \pi_H^* v_H.$$

Now, by integrating over $G$ we get

$$\begin{aligned}\int_G i_H(S)\mathrm{Vol}(\pi_{\mathrm{H}}(\mathrm{S}))\mathrm{dH} &\geq \int_G dH \int_S |\theta_H(x)|v_S \\ &= \int_S \left(\int_G |\theta_H(x)|dH\right) v_S \\ &= I(G)\mathrm{Vol}(S), \end{aligned} \tag{2.14}$$

where $I(G) := \int_G |\theta_H(x)|dH$. The last equality comes from the fact that $I(G)$ does not depend on the point $x$ (see [12, page 101]). We also have

$$\begin{aligned}\int_G i_H(S)\mathrm{Vol}(\pi_{\mathrm{H}}(\mathrm{S}))\mathrm{dH} &\leq \sup_H \mathrm{Vol}(\pi_H(S))\bar{\imath}(S) \\ &\leq 2\mathrm{Vol}(\pi_{H_0}(S))\bar{\imath}(S), \end{aligned} \tag{2.15}$$

where $H_0$ is an $m$-plane such that $2\mathrm{Vol}(\pi_{H_0}(S)) \geq \sup_H \mathrm{Vol}(\pi_H(S))$. By Inequalities (2.14) and (2.15), we get the following inequality

$$\mathrm{Vol}(\pi_{H_0}(S)) \geq \frac{I(G)\mathrm{Vol}(S)}{2\bar{\imath}(S)}.$$

This proves Inequality (2.13) with $C_m = \frac{2}{I(G)}$. □

Let $M$ be an $m$-dimensional immersed submanifold of $\mathbb{R}^{m+p}$. Throughout the rest of this section, for every $\varepsilon \geq 0$, $M_\varepsilon^D$ stands for $M \setminus D$, where $D$ is any open subdomain of $M$ such that $M \setminus D$ is a smooth manifold with smooth boundary and $\mathrm{Vol}(D) = \varepsilon \mathrm{Vol}(M)$.

**Corollary 2.3.1.** *For all $x \in \mathbb{R}^{m+p}$ and $\varepsilon \geq 0$, we have*

$$Vol\big(M_\varepsilon^D \cap B(x,s)\big) \leq \frac{2\,Vol(B^m)}{I(G)}\bar{\imath}_r(M_\varepsilon^D)s^m, \quad \forall\ 0 < s \leq r; \tag{2.16}$$

$$Vol\big(M_\varepsilon^D \cap B(x,r)\big) \leq \frac{2\,Vol(B^m)}{I(G)}\bar{\imath}(M_\varepsilon^D)r^m, \quad \forall r > 0, \tag{2.17}$$

*where $B^m$ is the $m$-dimensional Euclidean unit ball.*

*Proof.* Replacing $S$ by $M_\varepsilon^D \cap B(x,s)$ in Lemma 2.3.1, we obtain

$$\begin{aligned}\mathrm{Vol}\left(M_\varepsilon^D \cap B(x,s)\right) &\leq \frac{2}{I(G)}\bar{\imath}\left(M_\varepsilon^D \cap B(x,s)\right)\mathrm{Vol}\left(\pi_H\left(M_\varepsilon^D \cap B(x,s)\right)\right) \\ &\leq \frac{2\mathrm{Vol}(B^m)}{I(G)}\bar{\imath}_s\left(M_\varepsilon^D\right)s^m,\end{aligned}$$

where $B^m$ is the $m$-dimensional Euclidean unit ball.
The last inequality comes from

$$\mathrm{Vol}\left(\pi_{H_0}\left(M_\varepsilon^D \cap B(x,s)\right)\right) \leq \mathrm{Vol}\left(\pi_{H_0}\left(B(x,s)\right)\right) \leq \mathrm{Vol}(B^m)s^m$$

Since $\bar{\imath}_s\left(M_\varepsilon^D\right) \leq \bar{\imath}_r\left(M_\varepsilon^D\right)$ for all $0 < s \leq r$ and $\bar{\imath}_s\left(M_\varepsilon^D\right) \leq \bar{\imath}\left(M_\varepsilon^D\right)$ for all $s > 0$, therefore, we derive Inequalities (2.16) and (2.17). □

**Remark 2.3.1.** *For $\varepsilon = 0$, we have $M_\varepsilon^D = M$. Hence, we have the Inequalities (2.16) and (2.17) for $M_\varepsilon^D$ replaced by $M$.*

*Proof of Theorem 2.1.1.* This theorem is a straightforward consequence of Corollary 2.2.1. We begin with giving candidates for a distance $d$ and a measure $\mu$ such that the assumptions of Corollary 2.2.1 are satisfied. Let $d = d_{eu}$ be the Euclidean distance in $\mathbb{R}^{m+p}$ and $\mu = \mu_\epsilon^D$ where $\mu_\epsilon^D(A)$ is the Riemannian volume of $A \cap M_\varepsilon^D$. One can easily check that $(M, d_{eu})$ has the $(2,N)$-covering property where $N$ depends only on the dimension of the ambient space $\mathbb{R}^{m+p}$. Moreover, one can consider it as a function depending only on the dimension $m$ (see [12, page 106]). There also exists $L > 0$ such that $\mu_\varepsilon^D(B(x,s)) \leq Ls^m$ for $s \leq \rho$. Indeed, we consider two cases:

- Take $\rho = r$. According to Corollary 2.3.1, one can take $L = \frac{2\mathrm{Vol}(B^m)}{I(G)}\bar{\imath}_r(M_\varepsilon^D)$. Therefore, Corollary 2.2.1 implies

$$\lambda_k(M) \leq \alpha_m \frac{1}{r^2} + \beta_m \frac{\bar{\imath}_r(M_\varepsilon^D)^{2/m}}{(1-\varepsilon)^{1+2/m}}\left(\frac{k}{\mathrm{Vol}(M)}\right)^{2/m}. \tag{2.18}$$

- Take $\rho = \infty$. According to Corollary 2.3.1, one can take $L = \frac{2\mathrm{Vol}(B^m)}{I(G)}\bar{\imath}(M_\varepsilon^D)$. Therefore, Corollary 2.2.1 implies

$$\lambda_k(M) \leq \beta_m \frac{\bar{\imath}(M_\varepsilon^D)^{2/m}}{(1-\varepsilon)^{1+2/m}}\left(\frac{k}{\mathrm{Vol}(M)}\right)^{2/m}. \tag{2.19}$$

The left hand-sides of Inequalities (2.18) and (2.19) do not depend on $D$. Hence, taking the infimum over $D$, we get Inequalities (2.3) and (2.4). □

## 2.4 Eigenvalues of Complex Submanifolds of $\mathbb{C}P^N$

In this section, we provide the proof of Theorem 2.1.2. Before going into the proof we need to recall the universal inequality proved by El Soufi, Harrell and Ilias which is the key idea of the proof. The following lemma is a special case of that universal inequality [17, Theorem 3.1] (see also [9]):

**Lemma 2.4.1.** *Let $M^m$ be a compact complex manifold of complex dimension $m$ and $\phi : M \to \mathbb{C}P^N$ be a holomorphic immersion. Then the eigenvalues of the Laplace-Beltrami operator on $(M, \phi^* g_{FS})$ satisfy the following inequality:*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{2}{m}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i + c_m), \tag{2.20}$$

*where* $c_m = 2m(m+1)$.

Another useful result is the following recursion formula given by Cheng and Yang:

**Lemma 2.4.2.** *([8, Corollary 2.1]) If a positive sequence of numbers* $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{k+1}$*, satisfies the following inequality*

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}\mu_i(\mu_{k+1}-\mu_i), \tag{2.21}$$

*then*

$$\mu_{k+1} \leq (1+\frac{4}{n})k^{2/n}\mu_1.$$

**Theorem 2.4.1.** *Let* $M^m$ *be a compact complex manifold of complex dimension* $m$ *admitting a holomorphic immersion* $\phi : M \to \mathbb{C}P^N$*. Then for every* $k \in \mathbb{N}^*$ *we have*

$$\lambda_{k+1}(M, \phi^* g_{FS}) \leq 2(m+1)(m+2)k^{\frac{1}{m}} - 2m(m+1). \tag{2.22}$$

*Proof of Theorem 2.4.1.* According to Lemma 2.4.1, the eigenvalues of the Laplace operator on $M$ satisfy universal Inequality (2.20). We replace $\lambda_i$ by $\mu_i := \lambda_i + c_m$ in Inequality (2.20) and we obtain:

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2 \leq \frac{2}{m}\sum_{i=1}^{k}\mu_i(\mu_{k+1}-\mu_i).$$

One now has a positive sequence of numbers $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{k+1}$ that satisfies Inequality (2.21) with $n = 2m$. Applying the recursion formula of Cheng and Yang, we get the following inequality:

$$\mu_{k+1} \leq (1+\frac{4}{2m})k^{2/2m}\mu_1. \tag{2.23}$$

By replacing $\mu_i$ by $\lambda_i + c_m$ in Inequality (2.23), we obtain:

$$\lambda_{k+1}(M, \phi^* g_{FS}) \leq (1+\frac{2}{m})(\lambda_1(M, \phi^* g_{FS}) + c_m)k^{1/m} - c_m.$$

Since $M$ is a compact manifold, $\lambda_1(M, \phi^*(g_{FS}) = 0$. Therefore,

$$\lambda_{k+1}(M, \phi^* g_{FS}) \leq (1+\frac{2}{m})c_m k^{1/m} - c_m = 2(m+1)(m+2)k^{1/m} - 2m(m+1),$$

which completes the proof. □

As we mentioned in the introduction, for $k = 1$ we get a sharp upper bound:

$$\lambda_2(M, \phi^* g_{FS}) \leq \lambda_2(\mathbb{C}P^m, g_{FS}) = 4(m+1). \tag{2.24}$$

In [4], Bourguignon, Li and Yau obtained an upper bound for the first non-zero eigenvalue of a complex manifold $(M, \omega)$ which admits a *full* holomorphic immersion (i.e. $\Phi(M)$ is not contained in any hyperplane of $\mathbb{C}P^N$ ) into $\mathbb{C}P^N$ as following:

$$\lambda_2(M, \omega) \leq 4m\frac{N+1}{N}d([\Phi], [\omega]). \tag{2.25}$$

Here, $d([\Phi],[\omega])$ is the *holomorphic immersion degree* – a homological invariant – defined by

$$d([\Phi],[\omega]) = \frac{\int_M \Phi^*(\omega_{FS}) \wedge \omega^{m-1}}{\int_M \omega^m},$$

where $\omega_{FS}$ is the Kähler form of $\mathbb{C}P^N$ with respect to the Fubini-Study metric and $\omega$ is Kähler form on $M$.
If one takes $\omega = \Phi^*(\omega_{FS})$, then $d([\Phi],[\omega]) = 1$ and we get Inequality (2.24) as a corollary of Inequality (2.25). Theorem 2.4.1 gives us another proof without assuming the immersion to be full.

For any full holomorphic immersion $\Phi$ of the surface $\Sigma_\gamma$ with genus $\gamma$ inequality 2.25 gives

$$\lambda_2(\Sigma_\gamma,\omega)\mathrm{Vol}(\Sigma_\gamma,\omega) \leq 4\frac{N+1}{N}\deg(\Phi(\Sigma_\gamma)).$$

# Chapitre 3

# Valeurs propres des opérateurs de Laplace perturbés

# Eigenvalues of perturbed Laplace operators

## 3.1 Introduction and statement of the results

Let $(M^m, g)$ be a compact Riemannian manifold of dimension $m$ and let $q$ be a continuous function on $M$. We study the eigenvalues of the Schrödinger operator $L := \Delta_g + q$ acting on functions on $(M, g)$. The eigenvalues of the Schrödinger operator $L$ constitute an increasing unbounded sequence of real numbers.

$$\lambda_1(\Delta_g + q) \leq \lambda_2(\Delta_g + q) \leq \cdots \leq \lambda_k(\Delta_g + q) \leq \cdots \nearrow \infty.$$

Here, we study how the eigenvalues of $L$ can be controlled in terms of geometric invariants of the manifold and integral quantities depending on the potential. Due to the min-max theorem, we have the following variational characterization of the eigenvalues of the Schrödinger operator:

$$\lambda_k(\Delta_g + q) = \min_{V_k} \max_{0 \neq f \in V_k} \frac{\int_M |\nabla_g f|^2 d\mu_g + \int_M f^2 q d\mu_g}{\int_M f^2 d\mu_g}, \tag{3.1}$$

where $V_k$ is a $k$-dimensional linear subspace of $H^1(M)$ and $\mu_g$ is the Riemannian measure corresponding to the metric $g$. Having this variational formula, it is easy to see that

$$\lambda_1(\Delta_g + q) \leq \frac{1}{\mu_g(M)} \int_M q d\mu_g.$$

For the second eigenvalue $\lambda_2(\Delta_g + q)$, an upper bound in terms of the mean value of the potential $q$ and a conformal invariant was obtained by El Soufi and Ilias [18, Theorem 2.2]:

$$\lambda_2(\Delta_g + q) \leq m \left( \frac{V_c([g])}{\mu_g(M)} \right)^{\frac{2}{m}} + \frac{\int_M q d\mu_g}{\mu_g(M)}, \tag{3.2}$$

where $V_c([g])$ is the conformal volume that is defined by Li and Yau [29] which only depends on the conformal class $[g]$ of metric $g$.
In particular, for a Riemannian surface $(\Sigma_\gamma, g)$ of genus $\gamma$, one obtains the following inequality as a consequence of Inequality (3.2):

$$\lambda_2(\Delta_g + q) \leq \frac{8\pi}{\mu_g(\Sigma_\gamma)} \left[ \frac{\gamma + 3}{2} \right] + \frac{\int_{\Sigma_\gamma} q d\mu_g}{\mu_g(\Sigma_\gamma)}, \tag{3.3}$$

where $[\frac{\gamma+3}{2}]$ is the integer part of $\frac{\gamma+3}{2}$.

Now the following interesting and natural question arises.

**Question.** *Can one control the eigenvalues of the Schrödinger operator $L$ in terms of the mean value of the potential $q$ and geometric invariants of $M$?*

This question was investigated by Grigori'yan, Netrusov and Yau [24]. They proved a general and abstract result that can be stated in the case of Schrödinger operators as follows: Given positive constants $N$ and $C_0$, assume that a compact Riemannian manifold $(M, g)$ has the $(2, N)$-covering property (i.e. each ball of radius $r$ can be covered by at most $N$ balls of radius $r/2$) and $\mu_g(B(x,r)) \leq C_0 r^2$ for every $x \in M$ and every $r > 0$. Then for every function $q \in C^0(M)$ we have [24, Theorem 1.2 (1.14)]:

$$\lambda_k(\Delta_g + q) \leq \frac{Ck + \delta^{-1}\int_M q^+ d\mu_g - \delta \int_M q^- d\mu_g}{\mu_g(M)}, \tag{3.4}$$

where $\delta \in (0,1)$ is a constant which depends only on $N$, $C > 0$ is a constant which depends on $N$ and $C_0$, and $q^{\pm} = \max\{|\pm q|, 0\}$.
Moreover, if $L$ is a positive definite operator [24, Theorem 5.15], then

$$\lambda_k(\Delta_g + q) \leq \frac{Ck + \int_M q d\mu_g}{\epsilon \mu_g(M)}, \tag{3.5}$$

where $\epsilon \in (0,1)$ depends only on $N$ and $C$ depends on $N$ and $C_0$.
In [24], it is conjectured that Inequality (3.5) is also true without assuming $L$ to be positive.

The above inequalities in dimension 2 have special feature as follows. Let $\Sigma_\gamma$ be a Riemannian surface of genus $\gamma$. Then for every Riemannian metric $g$ on $\Sigma_\gamma$ and every $q \in C^0(\Sigma_\gamma)$ we have [24, Theorem 5.4]:

$$\lambda_k(\Delta_g + q) \leq \frac{Q(\gamma+1)k + \delta^{-1}\int_{\Sigma_\gamma} q^+ d\mu_g - \delta \int_{\Sigma_\gamma} q^- d\mu_g}{\mu_g(\Sigma_\gamma)},$$

where $\delta \in (0,1)$ and $Q > 0$ are absolute constants.

These Inequalities (3.4) and (3.5) are not consistent with the Weyl law regarding to the power of $k$, except in dimension 2. In what follows we obtain upper bounds which generalize and improve the above inequalities without imposing any condition on the metric and which are asymptotically consistent with the Weyl law.

**Theorem 3.1.1.** *There exist positive constants $\alpha_m \in (0,1)$, $B_m$ and $C_m$ depending only on $m$ such that for every compact $m$-dimensional Riemannian manifold $(M^m, g)$, every potential $q \in C^0(M)$ and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_g + q) \leq \frac{\alpha_m^{-1}\int_M q^+ d\mu_g - \alpha_m \int_M q^- d\mu_g}{\mu_g(M)} + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}, \tag{3.6}$$

*where $V([g])$ is the min-conformal volume which is defined in Chapter 1.*

*In particular, when the potential $q$ is nonnegative one has*

$$\lambda_k(\Delta_g + q) \leq A_m \frac{1}{\mu_g(M)} \int_M q d\mu_g + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}},$$

*where $A_m = \alpha_m^{-1}$.*

We also obtain a conformal upper bound for positive Schrödinger operators which generalizes Inequality (3.5).

**Theorem 3.1.2.** *There exist constants $A_m > 1$, $B_m$ and $C_m$ depending only on $m$ such that if $L = \Delta_g + q$, $q \in C^0(M)$ is a positive operator then for every compact $m$-dimensional Riemannian manifold $(M^m, g)$ and every $k \in \mathbb{N}^*$ we have*

$$\lambda_k(\Delta_g + q) \leq A_m \frac{\int_M q d\mu_g}{\mu_g(M)} + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}. \tag{3.7}$$

Given Schrödinger operator $L = \Delta_g + q$, for every $\varepsilon > 0$, the Schrödinger operator $\tilde{L} = \Delta_g + q - \lambda_1(L) + \varepsilon$ is positive and $\lambda_k(\tilde{L}) = \lambda_k(L) - \lambda_1(L) + \varepsilon$. When $\varepsilon$ goes to zero, Theorem 3.1.1 leads to the following:

**Corollary 3.1.1.** *Under the assumptions of Theorem 3.1.1 we get*

$$\lambda_k(\Delta_g + q) \leq A_m \frac{1}{\mu_g(M)} \int_M q d\mu_g + (1 - A_m)\lambda_1(\Delta_g + q) + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}.$$

In the 2 dimensional case, for a Riemannian surface $\Sigma_\gamma$ of genus $\gamma$, thanks to the uniformization and Gauss-Bonnet theorems, one has $V([g]) \leq 4\pi\gamma$. Therefore, for Riemannian surfaces one has all the above inequalities replacing the min-conformal volume by the topological invariant $4\pi\gamma$.

**Corollary 3.1.2.** *There exist absolute constants $a \in (0,1)$, $A$ and $B$ such that, for every compact Riemannian surface $(\Sigma_\gamma, g)$ of genus $\gamma$, every potential $q \in C^0(M)$ and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_g + q)\mu_g(\Sigma_\gamma) \leq \int_{\Sigma_\gamma} (aq^+ - a^{-1}q^-) d\mu_g + A\gamma + Bk. \tag{3.8}$$

*And if $L$ is a positive operator then*

$$\lambda_k(\Delta_g + q)\mu_g(\Sigma_\gamma) \leq a \int_{\Sigma_\gamma} q d\mu_g + A\gamma + Bk.$$

An interesting application of Theorem 3.1.1 is the case of weighted Laplace operators. Indeed, given a Riemannian manifold $(M, g)$ and a function $\phi \in C^2(M)$, the corresponding weighted Laplace operator $\Delta_\phi$ is defined as follows.

$$\Delta_\phi = \Delta_g + \nabla_g \phi \cdot \nabla_g.$$

This operator is associated with the quadratic functional $\int_M |\nabla_g f|^2 e^{-\phi} d\mu_g$ i.e.

$$\int_M \Delta_\phi f h e^{-\phi} d\mu_g = \int_M \langle \nabla_g f, \nabla_g h \rangle e^{-\phi} d\mu_g.$$

This operator is an elliptic operator on $C_c^\infty(M) \subseteq L^2(e^{-\phi} d\mu_g)$ and can be extended to a self-adjoint operator with weighted measure $e^{-\phi} d\mu_g$. In this sense, it arises as a generalization of the Laplacian. The weighted Laplace operator $\Delta_\phi$ is also known as the diffusion operator or the Bakry–Émery Laplace operator which is used to study the diffusion process (see for instance, the pioneering work of Bakry and Émery [2] or the survey on this topic by Lott and Villani [31]). A Riemannian manifold $(M, g)$ with the density $e^{-\phi}$ is denoted by the triple $(M, g, \phi)$ and is called a Bakry–Émery manifold (see [32], [35]). The interplay between geometry of $M$ and the behavior of $\phi$ is mostly taken into account by means of new notion of curvature called the Bakry–Émery Ricci tensor that is defined as follows

$$\mathrm{Ricci}_\phi = \mathrm{Ricci}_g + \mathrm{Hess}\phi.$$

Our aim is to find upper bounds for eigenvalues of $\Delta_\phi$ denoted by $\lambda_k(\Delta_\phi)$ in terms of the geometry of $M$ and of properties of $\phi$.
Upper bounds for the first eigenvalue $\lambda_1(\Delta_\phi)$ of a complete Bakry–Émery manifold have been recently considered in several works (see [33],[36],[37], [41] and [42]). These upper bounds depend on the $L^\infty$-norm of $\nabla_g\phi$ and a lower bound of the Bakry–Émery Ricci tensor:
Let $(M^m, g, \phi)$ be a complete Bakry–Émery manifold with $\mathrm{Ricci}_\phi \geq -\kappa^2(m-1)$ and $|\nabla_g\phi| \leq \sigma$ for some constants $\kappa \geq 0$ and $\sigma > 0$. Then we have [37, Proposition 2.1] (see also [33], [41] and [42]):

$$\lambda_1(\Delta_\phi) \leq \frac{1}{4}((m-1)\kappa + \sigma)^2. \tag{3.9}$$

In particular, if $\mathrm{Ricci}_\phi \geq 0$, then we have

$$\lambda_1(\Delta_\phi) \leq \frac{1}{4}\sigma^2. \tag{3.10}$$

We present two approaches to obtain upper bounds for the eigenvalues of the Bakry–Émery Laplace operator in terms of the geometry of $M$ and of the properties of $\phi$.
**First approach.** One can see that $\Delta_\phi$ is unitarily equivalent to the Schrödinger operator $L = \Delta + \frac{1}{2}\Delta_g\phi + \frac{1}{4}|\nabla_g\phi|^2$ (see for example [36, page 28]). Therefore, as a consequence of Theorem 3.1.1 we obtain an upper bound for $\lambda_k(\Delta_\phi)$, in terms of the geometry of $M$ and the $L^2$-norm of $\nabla_g\phi$.

**Theorem 3.1.3.** *There exist constants $A_m$, $B_m$ and $C_m$ depending on $m \in \mathbb{N}^*$, such that for every $m$-dimensional compact Riemannian manifold $(M, g)$, every $\phi \in C^2(M)$ and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_\phi) \leq A_m \frac{1}{\mu_g(M)} \|\nabla_g\phi\|^2_{L^2(M)} + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}.$$

It is worth noticing that in full generality, it is not possible to obtain upper bounds which do not depend on $\phi$ (see for instance [37, Section 2]). However, we will see that for compact Bakry–Émery manifolds with nonnegative Bakry–Émery Ricci curvature we can find upper bounds which do not depend on $\phi$ (see Corollary 3.1.3 below).
**Second approch.** Using the techniques that we successfully applied for the Laplace operator $\Delta_g$ on Riemannian manifolds in Chapter 1, we obtain upper bounds for eigenvalues of $\Delta_\phi$ in terms of a conformal invariant. We also obtain Buser's type inequality for $\lambda_k(\Delta_\phi)$.

**Definition 3.1.1.** *Let $(M, g, \phi)$ be a compact Bakry–Émery manifold. We define the $\phi-$min conformal volume as*

$$V_\phi([g]) = \inf\{\mu_\phi(M, g_0) : g_0 \in [g], \mathrm{Ricci}_\phi(M, g_0) \geq -(m-1)\}. \tag{3.11}$$

Up to dilations[1] there is always a Riemannian metric $g_0$ such that $\mathrm{Ricci}_\phi(M, g_0) \geq -(m-1)$. Before stating our result we need to introduce one more notation. Let $x \in M$ and $\phi \in C^2(M)$. Using geodesic polar coordinates centered at $x$, we denote the radial derivative of $\phi$ as $\partial_r\phi$. The inequality $\partial_r\phi > -\sigma$ means that for every $x \in M$ the corresponding radial gradient of $\phi$ is at least equal to $-\sigma$, where $\sigma \geq 0$.
We are now ready to state our theorem.

**Theorem 3.1.4.** *There exist positive constants $A(m)$ and $B(m)$ depending only on $m \in \mathbb{N}^*$ such that for every compact Bakry–Émery manifold $(M, g, \phi)$ with $\partial_r\phi \geq -\sigma$ for some $\sigma \geq 0$ and for every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_\phi) \leq A(m) \max\{\sigma^2, 1\} \left(\frac{V_\phi([g])}{\mu_\phi(M, g)}\right)^{2/m} + B(m) \left(\frac{k}{\mu_\phi(M)}\right)^{2/m}. \tag{3.12}$$

If a metric $g$ is conformally equivalent to a metric $g_0$ with $\mathrm{Ricci}_\phi(M, g_0) \geq 0$, then $V_\phi([g]) = 0$. Therefore, an immediate consequence of Theorem 3.1.4 is the following.

**Corollary 3.1.3.** *There exists a positive constant $A(m)$ which depends only on $m \in \mathbb{N}^*$ such that for every compact Bakry–Émery manifold $(M, g, \phi)$ with $V_\phi([g]) = 0$, and for every $k \in \mathbb{N}^*$*

$$\lambda_k(\Delta_\phi) \leq A(m) \left(\frac{k}{\mu_\phi(M)}\right)^{2/m}. \tag{3.13}$$

One can compare the above inequality with Inequality (3.10) above and with Inequality (1.8) for $\lambda_k$ in the Riemannian case with $\mathrm{Ricci}_g(M) \geq 0$ in Chapter 1.
If $\mathrm{Ricci}_\phi(M) > -\kappa^2(m-1)$ for some $\kappa \geq 0$, then for $g_0 = \kappa^2 g$ one has $\mathrm{Ricci}_\phi(M, g_0) > -(m-1)$ and $V_\phi([g]) \leq \mu_\phi(M, g_0) = \kappa^m \mu_\phi(M, g)$. Replacing in inequality 3.12 we get the following

**Corollary 3.1.4.** *There are positive constants $A(m)$ and $B(m)$ depending only on $m \in \mathbb{N}^*$ such that for every compact Bakry–Émery manifold $(M, g, \phi)$ with $\mathrm{Ricci}_\phi(M) > -\kappa^2(m-1)$ and $\partial_r\phi \geq -\sigma$ for some $\kappa \geq 0$ and $\sigma \geq 0$, and for every $k \in \mathbb{N}^*$*

$$\lambda_k(\Delta_\phi) \leq A(m) \max\{\sigma^2, 1\}\kappa^2 + B(m) \left(\frac{k}{\mu_\phi(M)}\right)^{2/m}.$$

[1]Notice Hess$\phi$ and $\mathrm{Ricci}_g$ do not change under dilations. If $\mathrm{Ricci}_\phi(M, g) \geq -\kappa^2(m-1)g$, then $\forall\alpha > 0$, $\mathrm{Ricci}_\phi(M, g_0) := \mathrm{Ricci}_\phi(M, \alpha g) = \mathrm{Ricci}_\phi(M, g) \geq -\kappa^2(m-1)g = -\frac{\kappa^2}{\alpha}(m-1)g_0$.

Corollary 3.1.4 can be proved directly by the classic idea used by Buser [6], Li and Yau [28]. We refer reader to the appendix where we give a simple direct proof.

**Remark 3.1.1.** *Notice that all of the results which mentioned above for compact manifolds, are also valid when bounded sudomains of complete manifolds with the Neumann boundary condition are considered.*

## 3.2 Eigenvalues of Schrödinger operators

In this section we prove Theorems 3.1.1 and 3.1.2 for Schrödinger operators. The idea of the proof is to construct a suitable family of test functions to be used in the variational characterization (3.1) of the eigenvalues of the Schrödinger operator $L$. According to this variational formula, for every family $\{f_i\}_{1=1}^k$ of disjointly supported test functions one has

$$\lambda_k(\Delta_g + q) \leq \max_{i\in\{1,\dots,k\}} \frac{\int_M |\nabla_g f_i|^2 d\mu_g + \int_M f_i^2 q d\mu_g}{\int_M f_i^2 d\mu_g}. \tag{3.14}$$

The technique that we introduced in the first chapter is the key method to construct a family of plateau functions supported on disjoint capacitors. We recall that a capacitor is a couple of open sets $(F, G)$ in $M$ such that $F \subsetneq G$. For each capacitor in a Riemannian manifold, we define the capacity and the $m$-capacity by (see Definition 1.3.1):

$$\operatorname{cap}_g(F,G) = \inf_{\varphi\in\mathcal{T}} \int_M |\nabla_g\varphi|^2 d\mu_g, \quad \text{and} \quad \operatorname{cap}_{[g]}^{(m)}(F,G) = \inf_{\varphi\in\mathcal{T}} \int_M |\nabla_g\varphi|^m d\mu_g, \tag{3.15}$$

where $\mathcal{T} = \mathcal{T}(F,G)$ is the set of all functions $\varphi \in C_0^\infty(M)$ such that supp $\varphi \subset G$ and $\varphi \equiv 1$ in a neighborhood of $F$. If $\mathcal{T}(F,G)$ is empty, then $\operatorname{cap}_g(F,G) = \operatorname{cap}_{[g]}^{(m)}(F,G) = +\infty$.
The following lemma follows from the technique we introduced in the first chapter.

**Lemma 3.2.1.** *Let $(M^m, g, \mu)$ be a compact Riemannian manifold with a non-atomic Borel measure $\mu$. Then there exist positive constants $c(m) \in (0,1)$ and $\alpha(m)$ depending only on the dimension such that for every $k \in \mathbb{N}^*$ there exists a family $\{(F_i, G_i)\}_{i=1}^k$ of capacitors with the following properties:*

*(I)* $\mu(F_i) > c(m)\frac{\mu(M)}{k}$,

*(II)* $\operatorname{cap}_g(F_i, G_i) \leq \frac{\mu_g(M)}{k}\left[\frac{1}{r_0^2}\left(\frac{V([g])}{\mu_g(M)}\right)^{2/m} + \alpha(m)\left(\frac{k}{\mu_g(M)}\right)^{2/m}\right]$.

*where* $r_0 = \frac{1}{1600}$.

*Proof.* Take the $m-m$ space $(M, d_{g_0}, \mu)$, where $g_0 \in [g]$ and $\text{Ricci}_{g_0} \geq -(m-1)$. According to Proposition 1.3.1, for every $k \in \mathbb{N}^*$ one has a family of $3k$ mutually disjoint capacitors $\{(F_i, G_i)\}_{i=1}^{3k}$, satisfying properties (i), (ii) and (iii) of Proposition 1.3.1. Hence, one obtains property (I) immediately from property (i) of Proposition 1.3.1.

For property (II), we first estimate $\operatorname{cap}_{[g]}^{(m)}(F_i, G_i)$. According to Proposition 1.3.1 the family $\{(F_i, G_i)\}_{i=1}^{3k}$ is such that either

(a) all the $F_i$'s are annuli, $G_i = 2F_i$ and $\mathrm{cap}^{(m)}_{[g_0]}(F_i, 2F_i) \leq Q_m$, or

(b) all the $F_i$'s are domains in $M$ and $G_i = F_i^{r_0}$ is the $r_0$-neighborhood of $F_i$.

In the second case, one can define a plateau function $\varphi_i \in \mathcal{T}(F_i, G_i)$ so that $|\nabla_{g_0}\varphi_i| \leq \frac{1}{r_0}$. Then

$$\mathrm{cap}^{(m)}_{[g]}(F_i, G_i) \leq \int_M |\nabla_{g_0}\varphi_i|^m d\mu_{g_0} \leq \frac{1}{r_0^m}\mu_{g_0}(G_i).$$

Since $G_1, \ldots, G_{3k}$ are mutually disjoint, there exist at least $2k$ of them so that $\mu_{g_0}(G_i) \leq \mu_{g_0}(M)/k$. Similarly, there exist at least $2k$ sets (not necessarily the same ones) such that $\mu_g(G_i) \leq \mu_g(M)/k$. Therefore, up to re-ordering, we assume that the first $k$ of them (i.e. $G_1, \ldots, G_k$) satisfy both the two following inequalities

$$\mu_g(G_i) \leq \mu_g(M)/k, \qquad \mu_{g_0}(G_i) \leq \mu_{g_0}(M)/k.$$

Hence, in both cases there exist $k$ capacitors $(F_i, G_i), 1 \leq i \leq k$ with

$$\mathrm{cap}^{(m)}_{[g]}(F_i, G_i) \leq Q_m + \frac{1}{r_0^m}\frac{\mu_{g_0}(M)}{k}.$$

For every $\varepsilon > 0$, we consider plateau functions $\{f_i\}_{i=1}^k$, $f_i \in \mathcal{T}(F_i, G_i)$ with

$$\int_M |\nabla_g f_i|^m d\mu_g \leq \mathrm{cap}^{(m)}_{[g]}(F_i, G_i) + \varepsilon.$$

Therefore,

$$\begin{aligned}
\mathrm{cap}_g(F_i, G_i) &\leq \int_M |\nabla_g f_i|^2 d\mu_g \leq \left(\int_M |\nabla_g f_i|^m d\mu_g\right)^{\frac{2}{m}} \left(\int_M 1_{\mathrm{supp} f_i} d\mu_g\right)^{1-\frac{2}{m}} \\
&= \left(\int_M |\nabla_{g_0} f_i|^m d\mu_{g_0}\right)^{\frac{2}{m}} \left(\int_M 1_{\mathrm{supp} f_i} d\mu_g\right)^{1-\frac{2}{m}} \\
&\leq \left(\mathrm{cap}^{(m)}_{[g]}(F_i, G_i) + \varepsilon\right)^{\frac{2}{m}} \mu_g(G_i)^{1-\frac{2}{m}} \\
&\leq \left(Q_m + \frac{1}{r_0^m}\frac{\mu_{g_0}(M)}{k} + \varepsilon\right)^{\frac{2}{m}} \mu_g(G_i)^{1-\frac{2}{m}} \\
&\leq \left[Q_m^{\frac{2}{m}} + \frac{1}{r_0^2}\left(\frac{\mu_{g_0}(M)}{k}\right)^{\frac{2}{m}} + \varepsilon^{\frac{2}{m}}\right]\left(\frac{\mu_g(M)}{k}\right)^{1-\frac{2}{m}}. \qquad (3.16)
\end{aligned}$$

where Inequality (3.16) is due to the well-know fact that

$$(a+b)^s \leq a^s + b^s$$

when $a, b$ are nonnegative real numbers and $0 < s \leq 1$. Letting $\varepsilon$ tends to zero and taking the infimum over $g_0 \in [g]$ with $\mathrm{Ricci}_{g_0} \geq -(m-1)$, we end the proof. $\square$

The potential $q$ in the Schrödinger operator is a signed function (notice that we can assume $q \in L^1(M)$ as well). We define a signed measure $\sigma$ associated to the potential $q$ by

$$\sigma(A) = \int_A q d\mu_g, \quad \text{for every measurable subset } A \text{ of } X.$$

For any signed measure $\nu$ we write $\nu = \nu^+ - \nu^-$, where $\nu^+$ and $\nu^-$ are the positive and negative parts of $\nu$, respectively. For any signed measure $\nu$ and $0 \leq \delta \leq 1$ we define a new signed measure $\nu_\delta$ as

$$\nu_\delta := \delta\nu^+ - \nu^-.$$

**Lemma 3.2.2.** *[24, Lemma 4.3] Let $\mu$ and $\nu$ be two signed measures on $M$. Then*

$$(\mu + \nu)_\delta \geq \mu_\delta + \nu_\delta.$$

**Proof of Theorem 3.1.1.** For a real number $\lambda \in \mathbb{R}$ define $\mu_\lambda := (\lambda\mu_g - \sigma)^+$ as a non-atomic Borel measure on $M$. We we apply Lemma 3.2.1 to $(M, g, \mu_\lambda)$. Thus, for every $k \in \mathbb{N}^*$ and every $\lambda \in \mathbb{R}$, there exists a family $\{(F_i, G_i)\}_{i=1}^{2k}$ of $2k$ capacitors satisfying the properties (I) and (II).
From now on, we take $\lambda = \lambda_k(L)$ and we have from property (I) of the lemma

$$(\lambda_k(L)\mu_g - \sigma)^+(F_i) \geq c(m)\frac{(\lambda_k(L)\mu_g - \sigma)^+(M)}{2k}.$$

The measure $(\lambda_k(L)\mu_g - \sigma)^-$ is also a non-atomic. Since $G_i$'s are mutually disjoint, there exist at least $k$ of them with measure not greater than $(\lambda_k(L)\mu_g - \sigma)^-(M)/k$. Up to reordering, assume that the first $k$ of them satisfy

$$(\lambda_k(L)\mu_g - \sigma)^-(G_i) \leq (\lambda_k(L)\mu_g - \sigma)^-(M)/k, \quad i \in \{1, \ldots, k\}.$$

Therefore

$$(\lambda_k(L)\mu_g - \sigma)^-(G_i) - (\lambda_k(L)\mu_g - \sigma)^+(F_i) \leq \frac{(\lambda_k(L)\mu_g - \sigma)^-(M)}{k} - c(m)\frac{(\lambda_k(L)\mu_g - \sigma)^+(M)}{2k} \tag{3.17}$$

For every $\epsilon > 0$ and every $1 \leq i \leq k$, we choose $f_i$ in $\mathcal{T}(F_i, G_i)$ such that:

$$\int_M |\nabla_g f_i|^2 d\mu_g \leq \mathrm{cap}_g(F_i, G_i) + \epsilon. \tag{3.18}$$

Inequality (3.14) implies that there exists $i \in \{1, \cdots, k\}$ so that

$$\lambda_k(L)\int_M f_i^2 d\mu_g \leq \int_M |\nabla_g f_i|^2 d\mu_g + \int_M f_i^2 q d\mu_g.$$

Hence, having Lemma 3.2.1 and Inequality (3.17) we get

$$\begin{aligned}
0 &\leq \int_M |\nabla_g f_i|^2 d\mu_g - \int_M f_i^2(\lambda_k(L) - q)d\mu_g \leq \mathrm{cap}_g(F_i, G_i) + \epsilon - \int_M f_i^2(\lambda_k - q)d\mu_g \\
&\leq \frac{\mu_g(M)}{2k}\left[\frac{1}{r_0^2}\left(\frac{V([g])}{\mu_g(M)}\right)^{2/m} + \alpha(m)\left(\frac{2k}{\mu_g(M)}\right)^{2/m}\right] + \epsilon \\
&\quad + \int_M f_i^2(\lambda_k(L) - q)^- d\mu_g - \int_M f_i^2(\lambda_k(L) - q)^+ d\mu_g \\
&\leq \frac{\mu_g(M)}{2k}\left[\frac{1}{r_0^2}\left(\frac{V([g])}{\mu_g(M)}\right)^{2/m} + \alpha(m)\left(\frac{2k}{\mu_g(M)}\right)^{2/m}\right] + \epsilon \qquad (3.19) \\
&\quad + \frac{(\lambda_k(L)\mu_g - \sigma)^-(M)}{k} - c(m)\frac{(\lambda_k(L)\mu_g - \sigma)^+(M)}{2k}. \qquad (3.20)
\end{aligned}$$

We now estimate the last two terms of the above inequality considering two alternatives:

- If $\lambda_k(L)$ is positive, then applying Lemma 3.2.2 for the measure $\lambda_k(L)\mu_g$ and signed measure $-\sigma$ with $\delta = \frac{c(m)}{2}$, we get

$$\frac{c(m)}{2}(\lambda_k(L)\mu_g - \sigma)^+(M) - (\lambda_k(L)\mu_g - \sigma)^-(M) \geq \frac{c(m)}{2}\sigma^-(M) - \sigma^+(M) + \frac{c(m)}{2}\lambda_k(L)\mu_g(M). \qquad (3.21)$$

  Replacing (3.21) in (3.20), and letting $\epsilon$ tends to zero gives the following

$$\lambda_k(\Delta_g + q) \leq \frac{\frac{2}{c(m)}\sigma^+(M) - \sigma^-(M)}{\mu_g(M)} + \frac{1}{c(m)r_0^2}\left(\frac{\mu_{g_0}(M)}{\mu_g(M)}\right)^{\frac{2}{m}} + \frac{2^{\frac{2}{m}}\alpha(m)}{c(m)}\left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}, \qquad (3.22)$$

- If $\lambda_k(L)$ is negative, then applying Lemma 3.2.2 for the signed measures $\lambda_k(L)\mu_g$ and $-\sigma$ with $\delta = \frac{c(m)}{2}$, implies

$$\frac{c(m)}{2}(\lambda_k(L)\mu_g - \sigma)^+(M) - (\lambda_k(L)\mu_g - \sigma)^-(M) \geq \frac{c(m)}{2}\sigma^-(M) - \sigma^+(M) + \lambda_k(L)\mu_g(M). \qquad (3.23)$$

  Replacing (3.23) in (3.20) and letting $\epsilon$ goes to zero gives the following

$$\lambda_k(\Delta_g + q) \leq \frac{\sigma^+(M) - \frac{c(m)}{2}\sigma^-(M)}{\mu_g(M)} + \frac{1}{2r_0^2}\left(\frac{\mu_{g_0}(M)}{\mu_g(M)}\right)^{\frac{2}{m}} + 2^{\frac{2}{m}-1}\alpha(m)\left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}. \qquad (3.24)$$

Therefore $\lambda_k(L)(M,g)$ is smaller than the sum of the right-hand sides of Inequalities (3.22) and (3.24). We finally obtain Inequality (3.6) with, for example, $\alpha_m = \frac{4}{c(m)}$. □

**Proof of Theorem 3.1.2 .** Let $(M, g, \mu_g)$ be an $m - m$ space. By Lemma 3.2.1, for every $k \in N^*$ there is a family of $2k$ disjoint capacitors $\{(F_i, G_i)\}_{i=1}^{2k}$ that satisfies the properties (I) and (II). We claim that one can find a family $\{f_i\}_{i=1}^{2k}$ of test functions with

$2f_i \in \mathcal{T}(F_i, G_i)$ and so that

$$\sum_{i=1}^{2k} \int_M f_i^2 q d\mu_g \leq \sum_{i=1}^{2k} \int_M |\nabla_g f_i|^2 d\mu_g + \int_M q d\mu_g. \tag{3.25}$$

If we have Inequality (3.25) then

$$\begin{aligned} \sum_{i=1}^{2k} \int_M \left(|\nabla_g f_i|^2 + f_i^2 q\right) d\mu_g &\leq 2\sum_{i=1}^{2k} \int_M |\nabla_g f_i|^2 d\mu_g + \int_M q d\mu_g \\ &\leq \frac{k}{2} \max_i \operatorname{cap}_g(F_i, G_i) + \int_M q d\mu_g. \end{aligned}$$

By assumption $\int_M \left(|\nabla_g f_i|^2 + f_i^2 q\right) d\mu_g$ is positive for each $1 \leq i \leq 2k$. Therefore, at least $k$ of them satisfy the following inequality (up to reordering we assume that the first $k$ of them satisfy the inequality):

$$\int_M |\nabla_g f_i|^2 + f_i^2 q d\mu_g \leq \frac{k \max_i \operatorname{cap}_g(F_i, G_i) + 2\int_M q d\mu_g}{2k}. \tag{3.26}$$

Inequality (3.26) together with the bounds of $\operatorname{cap}_g(F_i, G_i)$ and $\mu_g(F_i)$ given in Lemma 3.2.1 (I) and (II) lead to

$$\begin{aligned} \lambda_k(L) &\leq \max_i \frac{\int_M |\nabla_g f_i|^2 d\mu_g + \int_M f_i^2 q d\mu_g}{\int_M f_i^2 d\mu_g} \leq \frac{k \max_i \operatorname{cap}_g(F_i, G_i) + 2\int_M q d\mu_g}{2k\mu_g(F_i)} \\ &\leq \left( \frac{\mu_g(M)}{k} \left[ \frac{1}{r_0^2} \left( \frac{V([g])}{\mu_g(M)} \right)^{2/m} + \alpha(m) \left( \frac{k}{\mu_g(M)} \right)^{2/m} \right] + \frac{2}{k} \int_M q d\mu_g \right) \frac{4k}{c(m)\mu_g(M)}. \end{aligned}$$

This completes the proof of Theorem 3.1.2.

It remains to prove Inequality (3.25) which is proved in [24, Section 5]. For the reader's convenience we repeat the proof. We define the function $h$ by the following identity

$$\sum_{i=1}^{2k} f_i^2 + h^2 = 1. \tag{3.27}$$

Since $f_1, \ldots, f_{2k}$ are disjointly supported and $0 \leq f_i \leq \frac{1}{2}$, hence $h > \frac{1}{2}$. We now estimate the left-hand side of Inequality (3.25).

$$\int_M \left( \sum_{i=1}^{2k} f_i^2 + h^2 - h^2 \right) q d\mu_g = \int_M q d\mu_g - \int_M h^2 q d\mu_g \leq \int_M q d\mu_g + \int_M |\nabla_g h|^2 d\mu_g, \tag{3.28}$$

where the last inequality comes from the fact that the Schrödinger operator $L$ is positive. Identity (3.27) implies

$$-2h\nabla_g h = -\nabla_g h^2 = \sum_{i=1}^{2k} \nabla_g f_i^2 = 2\sum_{i=1}^{2k} f_i \nabla_g f_i.$$

Therefore,

$$|\nabla_g h|^2 \leq |2h\nabla_g h|^2 = \sum_{i=1}^{2k} |\nabla_g f_i^2|^2 = 4\sum_{i=1}^{2k} |f_i \nabla_g f_i|^2 \leq \sum_{i=1}^{2k} |\nabla_g f_i|^2. \tag{3.29}$$

Combining Inequalities (3.28) and (3.29) we get the Inequality (3.25). □

## 3.3 Eigenvalues of the Bakry–Émery Laplacian

In this section we consider eigenvalues of the Bakry–Émery Laplace operator $\Delta_\phi$ on a Bakry–Émery manifold $(M, g, \phi)$, where $M$ is a compact $m$-dimensional Riemannian manifold and $\phi \in C^2(M)$. We denote the weighted measure on $M$ by $\mu_\phi$ with

$$\mu_\phi(A) = \int_A e^{-\phi} d\mu_g, \quad \text{for all Borel subset } A \text{ of } M.$$

*Proof of Theorem 3.1.3.* As we mentioned in the introduction, one can see that $\Delta_\phi = \Delta_g + \nabla_g \phi \cdot \nabla_g$ is unitarily equivalent to the Schrödinger operator $L = \Delta_g + \frac{1}{2}\Delta_g \phi + \frac{1}{4}|\nabla_g \phi|^2$. Therefore, substituting in Inequality (3.7) of Theorem 3.1.1 we obtain

$$\lambda_k(L) \leq A_m \frac{1}{\mu_g(M)} \int_M \left(\frac{1}{2}\Delta_g \phi + \frac{1}{4}|\nabla_g \phi|^2\right) d\mu_g + B_m \left(\frac{V([g])}{\mu_g(M)}\right)^{\frac{2}{m}} + C_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}.$$

Stokes theorem implies that $\int_M \Delta_g \phi d\mu_g = 0$. This gives the result. □

Regarding Theorem 3.1.4, we need to use the characteristic variational formula for the Bakry–Émery Laplacian (see for example [32, Proposition 1] and [35, Proposition 4]).

$$\lambda_k(\Delta_\phi) = \inf_{\dim L = k} \sup_{f \in L} \frac{\int_M |\nabla_g f|^2 e^{-\phi} d\mu_g}{\int_M f^2 e^{-\phi} d\mu_g}, \tag{3.30}$$

where $L \subset H^1(M, \mu_\phi)$.

As in the previous section, the technique that we introduced in the first chapter is the key method to construct a family of plateau functions supported on disjoint capacitors. We define a capacitor and capacity in a Bakry–Émery manifold $(M, g, \phi)$ in an analogue way as before. A capacitor is a couple of open sets $(F, G)$ in $M$ such that $F \subsetneq G$. For each capacitor in a Bakry–Émery manifold $(M, g, \phi)$, we define the capacity and the $m$-capacity by:

$$\operatorname{cap}_\phi(F, G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^2 d\mu_\phi, \quad \text{and} \quad \operatorname{cap}_\phi^{(m)}(F, G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^m d\mu_\phi, \tag{3.31}$$

respectively, where $\mathcal{T} = \mathcal{T}(F, G)$ is the set of all functions $\varphi \in C_0^\infty(M)$ such that supp $\varphi \subset G$ and $\varphi \equiv 1$ in a neighborhood of $F$. If $\mathcal{T}(F, G)$ is empty, then $\operatorname{cap}_\phi(F, G) = \operatorname{cap}_\phi^{(m)}(F, G) = +\infty$.

Thanks to volume comparison theorem proved by Wei and Wylie [40] for Bakry–Émery manifolds, one can show that Bakry–Émery manifolds have local covering property (see Lemma 3.3.1 below).

**Theorem 3.3.1** ([40]). *Let $(M, g, \phi)$ be a compact Bakry–Émery manifold with Ricci$_\phi$[2] $\geq \alpha(m-1)$. If $\partial_r \phi \geq -\sigma$, with respect to geodesic polar coordinates centered at $x$, then for every $0 < r \leq R$ we have (assume $R \leq \pi/2\sqrt{\alpha}$ if $\alpha > 0$)*

$$\frac{\mu_\phi(B(x,R))}{\mu_\phi(B(x,r))} \leq e^{\sigma R} \frac{v(m,R,\alpha)}{v(m,r,\alpha)}, \tag{3.32}$$

*and in particular, let allow $r$ tends to zero, we get*

$$\mu_\phi(B(x,R)) \leq e^{\sigma R} v(m,R,\alpha), \tag{3.33}$$

*where $v(m,r,\alpha)$ is the volume of a ball of radius $r$ in the simply connected space form of constant sectional curvature $\alpha$.*

We proceed to the proof of Theorem 3.1.4 by a series of lemmas.

**Lemma 3.3.1.** *Let $(M, g, \phi)$ be a compact Bakry–Émery manifold with Ricci$_\phi \geq -\kappa^2(m-1)$ and $\partial_r \phi \geq -\sigma$ for some $\kappa \geq 0$ and $\sigma \geq 0$. There exist constants $N(m) \in \mathbb{N}^*$ and $\xi = \xi(\sigma, \kappa) > 0$ such that $(M, g, \phi)$ satisfies the $(2, N; \xi)$-covering property. Moreover, there exists positive a constant $C(m)$ such that for every $0 \leq r < R \leq \xi$ and $x \in M$, the annulus $A = A(x, r, R)$ satisfies $\mathrm{cap}_\phi^{(m)}(A, 2A)) \leq C(m)$.*

*Proof.* Take $\xi = min\{\frac{1}{\sigma}, \frac{1}{\kappa}\}$ (with $\xi = \infty$ if $\sigma = \kappa = 0$ ). We first show that $(M, \mu_\phi)$ has the doubling property for $r < 4\xi$, i.e.

$$\mu_\phi(B(x,r)) \leq C\mu_\phi(B(x,r/2)), \quad 0 < r < 4\xi,$$

for some positive constant $c$.
From this, it is easy to deduce that $(M, \mu_\phi)$ has the $(2, N; \xi)$-covering property for example with $N = c^4$. To prove the doubling property, we use the relative volume comparison theorem (3.32) as follows.

$$\frac{\mu_\phi(B(x,r))}{\mu_\phi(B(x,r/2))} \leq e^{\sigma r} \frac{v(m,r,-\kappa^2)}{v(m,r/2,-\kappa^2)} = e^{\sigma r} \frac{v(m,\kappa r,-1)}{v(m,\kappa r/2,-1)}.$$

Take $\tilde{r} := \kappa r$. Hence, for every $0 < r < 4\xi = 4\min\{\frac{1}{\sigma}, \frac{1}{\kappa}\}$, we get

$$\begin{aligned} e^{\sigma r} \frac{v(m,\kappa r,-1)}{v(m,\kappa r/2,-1)} &\leq e^4 \frac{v(m,\tilde{r},-1)}{v(m,\tilde{r}/2,-1)}; \quad 0 < \tilde{r} < 4, \\ &\leq \sup_{\tilde{r} \in (0,4)} e^4 \frac{v(m,\tilde{r},-1)}{v(m,\tilde{r}/2,-1)} =: c(m). \end{aligned}$$

Thus,

$$\frac{\mu_\phi(B(x,r))}{\mu_\phi(B(x,r/2))} \leq c(m), \quad \text{for every } 0 < r < \xi.$$

[2] The Bakry–Émery Ricci tensor Ricci$_\phi$ also referred to as $\infty$-Bakry–Émery Ricci tensor.
We will denote Ricci$_\phi$ and Hess$\phi$ by Ricci$_\phi(M, g)$ and Hess$_g\phi$ wherever any confusion might occur.

Therefore, $(M, g, \phi)$ has $(2, N; \xi)$-covering property where $N = c^4(m)$.
Let $A = A(x, r, R)$ and let $f \in \mathcal{T}(A, 2A)$ be the function defined in page 26. Using the same argument as before (see page 26)) we get

$$\begin{aligned}
\mathrm{cap}_\phi^{(m)}(A, 2A) &\leq \int_M |\nabla_g f|^m d\mu_\phi \leq (\frac{2}{r})^m \mu_\phi(A(x, r/2, r)) + (\frac{1}{R})^m \mu_\phi(A(x, R, 2R)) \\
&\leq (\frac{2}{r})^m \mu_\phi(B(x, r)) + (\frac{1}{R})^m \mu_\phi(B(x, 2R)).
\end{aligned}$$

Having the comparison theorem (3.3.1), one gets

$$\begin{aligned}
\mathrm{cap}_\phi^{(m)}(A, 2A) &\leq \left(\frac{2}{r}\right)^m e^{\sigma r} v(m, r, -\kappa^2) + \left(\frac{1}{R}\right)^m e^{2\sigma R} v(m, 2R, -\kappa^2) \\
&= \left(\frac{2}{\kappa r}\right)^m e^{\sigma r} v(m, \kappa r, -1) + \left(\frac{1}{\kappa R}\right)^m e^{2\sigma R} v(m, 2\kappa R, -1).
\end{aligned}$$

Take $\tilde{r} := \kappa r$. Hence, for every $0 < r < R \leq 2\xi = 2\min\{\frac{1}{\sigma}, \frac{1}{\kappa}\}$, we get

$$\begin{aligned}
\mathrm{cap}_\phi^{(m)}(A, 2A) &\leq \left(\frac{2}{\tilde{r}}\right)^m e^2 v(m, \tilde{r}, -1) + \left(\frac{1}{\tilde{R}}\right)^m e^4 v(m, 2\tilde{R}, -1) \\
&\leq \sup_{\tilde{r}, \tilde{R} \in (0,2)} \left[\left(\frac{2}{\tilde{r}}\right)^m e^2 v(m, \tilde{r}, -1) + \left(\frac{1}{\tilde{R}}\right)^m e^4 v(m, 2\tilde{R}, -1)\right] =: C(m).
\end{aligned}$$

This completes the proof. □

**Lemma 3.3.2.** *Let $(M^m, g, \phi)$ be a compact Bakry–Émery manifold with $\partial\phi \geq -\sigma$ for some $\sigma \geq 0$. Then there exist positive constants $c(m) \in (0, 1)$ and $\alpha(m)$ depending only on the dimension such that for every $k \in \mathbb{N}^*$ there exists a family $\{(F_i, G_i)\}_{i=1}^k$ of capacitors with the following properties:*

*(I)* $\mu_\phi(F_i) > c(m)\frac{\mu_\phi(M)}{k}$,

*(II)* $\mathrm{cap}_\phi(F_i, G_i) \leq \frac{\mu_\phi(M)}{k}\left[\max\{\sigma^2, 1\}\left(\frac{V_\phi([g])}{\mu_\phi(M)}\right)^{2/m} + \alpha(m)\left(\frac{k}{\mu_\phi(M)}\right)^{2/m}\right]$.

*Proof.* Take the Bakry–Émery manifold $(M, g, \phi)$ as the $m - m$ space $(M, d_{g_0}, \mu_\phi)$ where $g_0 \in [g]$ with $\mathrm{Ricci}_\phi(M, g_0) \geq -(m-1)$ and $\mu_\phi$ is the weighted measure with respect to the metric $g$ (to avoid any confusion, we denote $\mu_\phi(A)$ by $\mu_\phi(A, g)$, $\forall A \subset M$ measurable subset, when different metrics are considered). According to Lemma 3.3.1, this space has the $(2, N, \xi)$-covering property with $\xi = \min\{\frac{1}{\sigma}, 1\}$. Hence, applying Theorem 1.2.1, for every $k \in \mathbb{N}^*$ one has a family of $3k$ mutually disjoint capacitors $\{F_i, G_i\}$ satisfying

$$\mu_\phi(F_i) \geq c(m)\frac{\mu_\phi(M)}{k},$$

where $c(m)$ is a positive constant depending only on the dimension. This immediately gives us the first property of Lemma 3.3.2 for $1 \leq i \leq k$.

For the second property, as in the proof of Lemma 3.2.1 we first estimate $\mathrm{cap}_\phi^{(m)}(F_i, G_i)$. Theorem 1.2.1 and Lemma 3.3.1 together imply that the family $\{(F_i, G_i)\}_{i=1}^{3k}$ is such that either

(a) all the $F_i$'s are annuli, $G_i = 2F_i$ with outer radii smaller than $\xi$ and $\mathrm{cap}_\phi^{(m)}(F_i, 2F_i) \leq C(m)$, or

(b) all the $F_i$'s are domains in $M$ and $G_i = F_i^{r_0}$ is the $r_0$-neighborhood of $F_i$ with $r_0 = \frac{\xi}{1600}$.

In the second case, one can define a plateau function $\varphi_i \in \mathcal{T}(F_i, G_i)$ so that $|\nabla_{g_0}\varphi_i| \leq \frac{1}{r_0}$. Then

$$\mathrm{cap}_\phi^{(m)}(F_i, G_i) \leq \int_M |\nabla_{g_0}\varphi_i|^m e^{-\phi} d\mu_{g_0} \leq \frac{1}{r_0^m}\mu_\phi(G_i, g_0).$$

Since $G_1, \ldots, G_{3k}$ are mutually disjoint, by the same reason as in Lemma 3.2.1 there exist at least $k$ of them, up to re-ordering, we assume that the first $k$ of them (i.e. $G_1, \ldots, G_k$) satisfy both the two following inequalities

$$\mu_\phi(G_i, g) \leq \mu_\phi(M, g)/k, \qquad \mu_\phi(G_i, g_0) \leq \mu_\phi(M, g_0)/k.$$

Hence, in both cases there exist $k$ capacitors $(F_i, G_i), 1 \leq i \leq k$ with

$$\mathrm{cap}_\phi^{(m)}(F_i, G_i) \leq Q_m + \frac{1}{r_0^m}\frac{\mu_\phi(M, g_0)}{k}.$$

For every $\varepsilon > 0$, we consider plateau functions $\{f_i\}_{i=1}^k$, $f_i \in \mathcal{T}(F_i, G_i)$ with

$$\int_M |\nabla_g f_i|^m e^{-\phi} d\mu_g \leq \mathrm{cap}_\phi^{(m)}(F_i, G_i) + \varepsilon.$$

Therefore, in the analogue way as before (see (3.16)) we obtain

$$\begin{aligned}
\mathrm{cap}_\phi(F_i, G_i) &\leq \int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g \leq \left(\int_M |\nabla_{g_0} f_i|^m e^{-\phi} d\mu_{g_0}\right)^{\frac{2}{m}} \left(\int_M 1_{\mathrm{supp} f_i} e^{-\phi} d\mu_g\right)^{1-\frac{2}{m}} \\
&\leq \left[C(m)^{\frac{2}{m}} + \frac{1}{r_0^2}\left(\frac{\mu_\phi(M, g_0)}{k}\right)^{\frac{2}{m}} + \varepsilon^{\frac{2}{m}}\right]\left(\frac{\mu_\phi(M, g)}{k}\right)^{1-\frac{2}{m}}.
\end{aligned}$$

Having $\frac{1}{r_0^2} = 1600\max\{\sigma^2, 1\}$ and letting $\varepsilon$ tends to zero and finally taking the infimum over $g_0 \in [g]$ with $\mathrm{Ricci}_\phi(M, g_0) \geq -(m-1)$, we end the proof. □

*Proof of Theorem 3.1.4.* According to Lemma 3.3.2 for $k \in \mathbb{N}^*$ we have a family of $k$ capacitors with the properties mentioned above. For every $\varepsilon > 0$, take $f_i \in \mathcal{T}(F_i, G_i)$, $1 \leq i \leq k$, so that

$$\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g \leq \mathrm{cap}_\phi(F_i, G_i) + \varepsilon.$$

Hence the characteristic variational formula (3.30) gives

$$\begin{aligned}\lambda_k(\Delta_\phi) &\leq \max_i \frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g}{\int_M f_i^2 e^{-\phi} d\mu_g} \leq \max_i \frac{\mathrm{cap}_\phi(F_i, G_i) + \varepsilon}{\mu_\phi(F_i)} \\ &\leq \frac{k}{c(m)\mu_\phi(M)} \left( \frac{\mu_\phi(M)}{k} \left[ \max\{\sigma^2, 1\} \left( \frac{V_\phi([g])}{\mu_\phi(M)} \right)^{2/m} + \alpha(m) \left( \frac{k}{\mu_\phi(M)} \right)^{2/m} \right] + \varepsilon \right).\end{aligned}$$

Letting $\varepsilon$ goes to zero, we get the desired inequality. $\square$

# Annexe

# Annexe A

# Buser type upper bound on Bakry-Émery manifolds

Here, we present a direct and simple proof of a weaker version of Corollary 3.1.4. This idea of proof was used by Buser [6, Satz 7], Cheng [10], Li and Yau [28] in the case of the Laplace–Beltrami operator. It is based on constructing a family of balls as capacitors which shall be the support of test functions. We can successfully apply this idea in the case of the Bakry–Émery Laplace operator.

**Theorem A.0.1** (Buser type inequality)**.** *Let $(M, g, \phi)$ be a compact Bakry–Émery manifold with $\mathrm{Ricci}_\phi(M) > -\kappa^2(m-1)$ and $\partial_r\phi \geq -\sigma$ for some $\kappa \geq 0$ and $\sigma \geq 0$. There are positive constants $A(m)$ and $B(m)$ such that for every $k \in \mathbb{N}^*$*

$$\lambda_k(\Delta_\phi) \leq A(m)\max\{\sigma, \kappa\}^2 + B(m)\left(\frac{k}{\mu_\phi(M)}\right)^{2/m}.$$

To see that this theorem is weaker than Corollary 3.1.4, consider the case where $\mathrm{Ricci}_\phi(M, g)$ is nonnegative. Indeed, the upper bound in Theorem A.0.1 still depends on $\sigma$ while Corollary 3.1.4 provids and estimate which depends only on the dimension.

*Proof.* Since $\mathrm{Ricci}_\phi(M) > -\kappa^2(m-1)$ and $\partial_r\phi \geq -\sigma$, comparison theorem gives us the following inequalities for every $0 < r \leq \xi = \min\{\frac{1}{\sigma}, \frac{1}{\kappa}\}$ (with $\xi = \infty$ if $\sigma = \kappa = 0$):

$$\frac{\mu_\phi(B(x,r))}{\mu_\phi(B(x,r/2))} \leq e^{\sigma r}\frac{v(m,r,-\kappa^2)}{v(m,r/2,-\kappa^2)} \leq \sup_{r\in(0,\xi)} e^{\sigma r}\frac{v(m,r,-\kappa^2)}{v(m,r/2,-\kappa^2)} =: c_1(m),$$

$$\mu_\phi(B(x,r)) \leq e^{\sigma r}v(m,r,-\kappa^2) \leq \sup_{s\in(0,\xi)} e^{\sigma s}v(m,s,-\kappa^2)r^m =: c_2(m)r^m. \tag{A.1}$$

Given $k \in \mathbb{N}^*$ let $\rho(k)$ be the positive number defined by

$$\rho(k) = \sup\{r : \exists p_1, \ldots, p_k \in M \text{ with } d_g(p_i, p_j) > r, \forall i \neq j\}.$$

We consider two alternatives:

Case 1. $\rho(k) \geq \xi$. For every $r < \xi$, there are $k$ points $p_1, \ldots, p_k$ with $B(p_i, r/2) \cap B(p_j, r/2) = \emptyset, \forall i \neq j$. For each $i \in \{1, \ldots, k\}$, we consider a plateau functions $f_i \in \mathcal{T}(B(p_i, r/4), B(p_i, r/2))$ defined as in (1.19). We get for every $1 \leq i \leq k$

$$\frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g}{\int_M f_i^2 e^{-\phi} d\mu_g} \leq \frac{16}{r^2} \frac{\mu_\phi(B(p_i, r/2))}{\mu_\phi(B(p_i, r/4))} \leq c_1(m) \frac{16}{r^2}.$$

Since this inequality is satisfied for every $r < \xi$, it also hold for $\xi$. Therefore

$$\frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g}{\int_M f_i^2 e^{-\phi} d\mu_g} \leq c_1(m) \frac{16}{\xi^2} \leq A(m) \max\{\sigma, \kappa\}^2.$$

Case 2. $\rho(k) < \xi$. Let us take $r < \rho(k)$ very close to $\rho(k)$. As in Case 1, there are $k$ points $p_1, \ldots, p_k$ with $B(p_i, r/2) \cap B(p_j, r/2) = \emptyset$, $\forall i \neq j$. Repeating the same argument we get for every $1 \leq i \leq k$

$$\frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g}{\int_M f_i^2 e^{-\phi} d\mu_g} \leq c_1(m) \frac{16}{r^2}.$$

Therefore, for every $1 \leq i \leq k$

$$\frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g}{\int_M f_i^2 e^{-\phi} d\mu_g} \leq c_1(m) \frac{16}{\rho(k)^2}.$$

We now estimate $\rho(k)$. Let $\rho(k) < s < \xi$ and $n$ be the maximal number of points $q_1, \ldots, q_n \in M$ so that $d(q_i, q_j) > s$, $\forall i \neq j$. Of course $n \leq k$ and because of the maximality of $n$, the balls $\{B(q_i, s)\}_{i=1}^n$ cover $M$. Hence, according to Inequality (A.1)

$$\mu_\phi(M) \leq \sum_{i=1}^n \mu_\phi(B(q_i, s)) \leq n c_2(m) s^m \leq k c_2(m) s^m.$$

Thus, letting $s$ tend to $\rho(k)$ we get

$$\frac{1}{\rho(k)^2} \leq c_2(m)^{2/m} \left( \frac{k}{\mu_\phi(M)} \right)^{2/m}.$$

Therefore,

$$\frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g}{\int_M f_i^2 e^{-\phi} d\mu_g} \leq 16 c_1(m) c_2(m)^{2/m} \left( \frac{k}{\mu_\phi(M)} \right)^{2/m}.$$

In conclusion we obtain

$$\lambda_k(\Delta_\phi) \leq \max_i \frac{\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g}{\int_M f_i^2 e^{-\phi} d\mu_g} \leq A(m) \max\{\sigma, \kappa\}^2 + B(m) \left( \frac{k}{\mu_\phi(M)} \right)^{2/m}.$$

This completes the proof. $\square$

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**Résumé :** Le but de cette thèse est de trouver des bornes supérieures pour les valeurs propres des opérateurs naturels agissant sur les fonctions d'une variété compacte $(M, g)$. Nous étudions l'opérateur de Laplace–Beltrami et des opérateurs du type laplacien. Dans le cas du laplacien, deux aspects sont étudiés. Le premier aspect est d'étudier des relations entre la géométrie intrinsèque et les valeurs propres du laplacien. Nous obtenons des bornes supérieures ne dépendant que de la dimension et d'un invariant conforme qui s'appelle le *volume conforme minimal*. Asymptotiquement, ces bornes sont consistantes avec la loi de Weyl. Elles améliorent également les résultats de Korevaar et de Yang et Yau. La méthode employée est intéressante en soi. Le deuxième aspect est d'étudier la relation entre la géométrie extrinsèque et les valeurs propres du laplacien agissant sur des sous-variétés compactes de $\mathbb{R}^N$ et de $\mathbb{C}P^N$. Nous étudions un invariant extrinsèque qui s'appele *l'indice d'intersection*. Pour des sous-variétés compactes de $\mathbb{R}^N$, nous généralisons les résultats de Colbois, Dryden et El Soufi et obtenons des bornes supérieures qui sont stables par des petites perturbations. Pour des sous-variétés de $\mathbb{C}P^N$, nous obtenons une borne supérieure ne dépendant que du *degré* des sous-variétés. Pour des opérateur du type laplacien, une modification de notre méthode donne des bornes supérieures pour les valeurs propres des opérateurs de Schrödinger en termes du volume conforme minimal et de l'intégrale du potentiel. Nous obtenons également les bornes supérieures pour les valeurs propres du laplacien de Bakry–Émery dépendant d'invariants conformes.

**Mots clés :** laplacien, opérateur de Schrödinger, laplacien de Barky-Émery, valeur propre, borne suppérieure, volume confrome minimal, nombre d'intersection moyenne.

**Abstract :** The purpose of this thesis is to find upper bounds for the eigenvalues of natural operators acting on functions on a compact Riemannian manifold $(M, g)$ such as the Laplace–Beltrami operator and Laplace-type operators. In the case of the Laplace–Beltrami operator, two aspects are investigated: The first aspect is to study relationships between the intrinsic geometry and eigenvalues of the Laplacian operator. In this regard, we obtain upper bounds depending only on the dimension and a conformal invariant called *min-conformal volume*. Asymptotically, these bounds are consistent with the Weyl law. They improve previous results by Korevaar and Yang and Yau. The method which is introduced to obtain the results, is powerful and interesting in itself. The second aspect is to study the interplay of the extrinsic geometry and eigenvalues of the Laplace–Beltrami operator acting on compact submanifolds of $\mathbb{R}^N$ and of $\mathbb{C}P^N$. We investigate an extrinsic invariant called the *intersection index* studied by Colbois, Dryden and El Soufi. For compact submanifolds of $\mathbb{R}^N$, we extend their results and obtain upper bounds which are stable under small perturbation. For compact submanifolds of $\mathbb{C}P^N$, we obtain an upper bound depending only on the *degree* of submanifolds. For Laplace type operators, a modification of our method lead to have upper bounds for the eigenvalues of Schrödinger operators in terms of the min-conformal volume and integral quantity of the potential. As another application of our method, we obtain upper bounds for the eigenvalues of the Bakry–Émery Laplace operator depending on conformal invariants.

**Keywords :** Laplacian, Schrödinger operator, Bakry–Émery Laplacian, eigenvalue, upper bound, min-conformal volume, mean intersection index.